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A CIRCLE REPRESENTATION USING COMPLEX AND QUATERNION NUMBERS

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Abstract. In this paper, we describe a simple representation of a circular arc in the space using quaternions. Using this representation, we obtain a subdivision of the arc, describe circular splines, and give a few applications with circular surfaces.

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1. INTRODUCTION

We start with the fractional linear function which produces a circle. Using this construction, we describe circular splines and subdivision of a circular arc. This approach is unusual, since we use the Bézier representation with complex (quaternion) points and complex (quaternion) weights. The conventional approach for representing a circular arc uses the control polygon of an isosceles triangle p_0, p_1, p_2 and the middle weight $w_1 = \cos\alpha$, where $\alpha = \angle(p_2 p_0 p_1)$ (see [3]). The calculation of cosine involves a computation of a square root, which is, in turn, often required for the representation of circular surfaces (a surface with one parameter family of circles). We describe an alternative approach and represent an arc of a circle by two end points and an initial (resp. final) tangent vector pointing into (resp. out) an arc. In the construction of an arc, we do not use the center of a circle and an angle. This simplifier calculations and is useful for representing families of circles on a circular surface (see applications in Section 8). There are many other circle representations (see [8]) but we hope that this approach can be useful for the subdivision of a circle and for modelling a circular surface.

2. NOTATIONS AND DEFINITIONS

We denote by $\mathbb{H}, \mathbb{C}$ and $\mathbb{R}$ the sets of quaternion numbers, complex numbers and real numbers, respectively. It is convenient to identify a complex number $z = \Re(z) + i \cdot \Im(z) = x + iy \in \mathbb{C}$ ($\Re(z)$ and $\Im(z)$ mean the real and imaginary parts of a complex number z) with the point (x, y) in the plane $\mathbb{R}^2$. The notation $\bar{z} = x - iy$ means the conjugated complex number and $|z| = \sqrt{x^2 + y^2}$ is the length of a complex number.

In general, the quaternion set $\mathbb{H}$ can be represented as

$$\mathbb{H} = \left\{ q = (r, p) \mid r \in \mathbb{R},\ \ p \in \mathbb{R}^3 \right\} = \mathbb{R}^4. \tag{1}$$

We denote the real and imaginary parts of a quaternion $q = (r, p)$ by $\Re(q) = r$ and $\Im(q) = p$. The multiplication in the algebra $\mathbb{H}$ is defined as

$$(r_1, p_1)(r_2, p_2) = \big(r_1 r_2 - \langle p_1, p_2 \rangle, (r_1 p_2 + r_2 p_1) + p_1 \times p_2\big), \tag{2}$$

where $\langle p_1, p_2 \rangle$ and $p_1 \times p_2$ are scalar and vector products in $\mathbb{R}^3$. We denote by $\bar{q} = (r, -p)$ the conjugate quaternion to $q = (r, p)$, $|q| = \sqrt{r^2 + \langle p, p \rangle} = \sqrt{q\bar{q}}$ is the length of the quaternion, $q^{-1} = \bar{q}/|q|^2 = (r/|q|^2, -p/|q|^2)$ denote the multiplicative inverse of q, i.e., $qq^{-1} = q^{-1}q = 1$. Denote the set of pure imaginary quaternions by

$$Im(\mathbb{H}) = \left\{ (0, p) \mid p \in \mathbb{R}^3 \right\} = \mathbb{R}^3. \tag{3}$$

Assume that $\langle u, u \rangle = 1$, $u \in \mathbb{R}^3$, then denote $\mathrm{e}^{\alpha u} = (\cos\alpha, (\sin\alpha)u)$. In fact, for any $q = (r, p)$ such that $|q|^2 = r^2 + \langle p, p \rangle = 1$, we have $q = \mathrm{e}^{\alpha u}$, where $\cos\alpha = r$, $u = p/\sin\alpha$.

3. LINEAR FRACTIONAL FUNCTION

PROPOSITION 1. *Let $a_0, a_1, w_0, w_1 \in \mathbb{C}$ be such that $w_0/w_1 \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\xi(t) = \frac{a_0 w_0 (1-t) + a_1 w_1 t}{w_0 (1-t) + w_1 t}, \quad t \in [0, 1], \tag{4}$$

is the arc of a circle with the first point a_0 and the last a_1. This circle has a center ξ_0 and the radius R, i.e., $|\xi(t) - \xi_0| = R$, where

$$\xi_0 = \frac{i(w_0 \bar{w}_1 a_0 - w_1 \bar{w}_0 a_1)}{2\Re(i w_0 \bar{w}_1)} = \frac{w_0 \bar{w}_1 a_0 - w_1 \bar{w}_0 a_1}{w_0 \bar{w}_1 - w_1 \bar{w}_0}, \tag{5}$$

$$R = |a_0 - \xi_0| = |a_1 - \xi_0| = \left| \frac{i(\bar{w}_0 w_1 (a_0 - a_1))}{2\Re(i w_0 \bar{w}_1)} \right|, \tag{6}$$

$$\frac{(a_1 - \xi_0)}{(a_0 - \xi_0)} = \frac{\bar{w}_0 w_1}{w_0 \bar{w}_1} = \mathrm{e}^{2\phi i} \quad \textit{(i.e., the circular arc angle is } 2\pi - 2\phi), \tag{7}$$

$$\cos\phi = \frac{w_1 \bar{w}_0 + w_0 \bar{w}_1}{2|w_0||w_1|}. \tag{8}$$

Proof. It is well known that the map $z \to \xi$, $\xi = \frac{az+b}{cz+d}$ for $z, \xi \in \mathbb{C}$ is conformal. The image of circles and lines are lines and circles (see [1], [5] Section 7.2). The formulas for ξ_0, R are taken from [1].

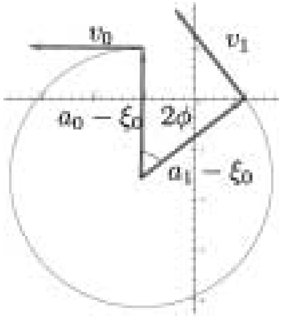

Figure 1. The arc of the circle: $c_0 = (-1+i, 1)$, $c_1 = (1, \mathrm{e}^{8i\pi/7})$, $\xi_0 = -1.03 - 1.57i$, $R = 2.57$.

We say that points $c_0 = (a_0 w_0, w_0), c_1 = (a_1 w_1, w_1)$ with weights w_0, w_1 define the arc (4) and denote it by $\xi(c_0, c_1)(t)$. We omit (c_0, c_1) if from the context it is clear (or not important) which points are taken.

Remark 1. If we change the weights w_0, w_1 to $1, w_1/w_0$, the arc $\xi(t)$ remains the same. Moreover, if we change the parameter t to $\rho t/(1 - t + \rho t)$ ($\rho \in \mathbb{R}$) and weights w_0, w_1 to $w_0, w_1/\rho$, then the arc also is the same. Therefore, we can always assume that the weights are normalized, i.e., $w_0' = 1, w_1' = \mathrm{e}^{i\phi} = \frac{w_1|w_0|}{w_0|w_1|}$. For example, if we take $w_0 = 1, w_1 = i$, then three points $a_0, \xi_0 = \frac{a_0+a_1}{2}, a_1$ are collinear and the radius is $R = \left|\frac{a_0-a_1}{2}\right|$.

Remark 2. Usually, the circle is uniquely defined by three points $a_0, b, a_1 \in \mathbb{C}$. If we set $w_1 = b - a_0, w_0 = a_1 - b$, then the arc (4) $\xi(t)$ goes through three points a_0, b, a_1 (in fact, $\xi(1/2) = b$).

PROPOSITION 2. *We have*

$$\xi(t) = \frac{a_0|w_0|^2(1-t)^2 + (a_0 w_0 \bar{w}_1 + a_1 w_1 \bar{w}_0)(1-t)t + a_1|w_1|^2 t^2}{|w_0|^2(1-t)^2 + (w_0\bar{w}_1 + w_1\bar{w}_0)(1-t)t + |w_1|^2 t^2}, \tag{9}$$

i.e., $\xi(t)$ is a rational quadratic Bézier curve with three control points and real weights

$$\left(a_0|w_0|^2, |w_0|^2\right), \quad \left(\frac{a_0 w_0 \bar{w}_1 + a_1 w_1 \bar{w}_0}{2}, \frac{w_0\bar{w}_1 + w_1\bar{w}_0}{2}\right), \quad \left(a_1|w_1|^2, |w_1|^2\right). \tag{10}$$

Proof. According to Remark 1, the arc $\xi(t)$ remains the same if we normalize the weights, i.e., $w_0' = 1, w_1' = \mathrm{e}^{i\phi} = \frac{w_1|w_0|}{w_0|w_1|}$. Therefore, we have

$$\begin{aligned}\xi(t) &= \frac{(a_0 w_0'(1-t) + a_1 w_1' t)(\bar{w}_0'(1-t) + \bar{w}_1' t)}{(w_0'(1-t) + w_1' t)(\bar{w}_0'(1-t) + \bar{w}_1' t)} \\ &= \frac{a_0(1-t)^2 + (a_0\mathrm{e}^{-\phi i} + a_1\mathrm{e}^{\phi i})(1-t)t + a_1 t^2}{(1-t)^2 + 2\cos(\phi)(1-t)t + t^2}.\end{aligned}$$

In the book [3], Chapter 14, we find that the last rational expression is a circle parametrization.

4. QUATERNION

For representing a circle in $\mathbb{R}^4$ or $\mathbb{R}^3$, we use the linear space of quaternion $\mathbb{H} = \mathbb{R}^4$. Let

$$\xi(t) = qd^{-1}, \quad \text{where } q = a_0 w_0 (1-t) + a_1 w_1 t, \tag{11}$$

$$d = w_0(1-t) + w_1 t, \quad a_0, a_1, w_0, w_1 \in \mathbb{H}, \quad t \in [0,1], \tag{12}$$

denote a linear fractional function in a quaternion algebra $\mathbb{H}$.

PROPOSITION 3. $\xi(t),\ t \in [0,1]$, *is an arc of a circle in* $\mathbb{R}^4$ *with end points* $\xi(0) = a_0, \xi(1) = a_1 \in \mathbb{H} = \mathbb{R}^4$. *This circle has a center* ξ_0, *a radius* R, *and* 2α *is the circular arc angle*

$$\xi_0 = \frac{a_0 + a_1 - 2w'^2 p_1}{2(1 - w'^2)}, \quad R = \frac{|a_0 - a_1|}{2\sqrt{1 - w'^2}}, \quad \textit{where} \tag{13}$$

$$p_1 = \frac{a_0 w_0 \bar{w}_1 + a_1 w_1 \bar{w}_0}{w_0 \bar{w}_1 + w_1 \bar{w}_0}, \quad w' = (w_0 \bar{w}_1 + w_1 \bar{w}_0)/(2|w_0||w_1|) = \cos\alpha. \tag{14}$$

Moreover, $\xi(t)$ *is a rational quadratic Bézier curve which has three control points with real weights*

$$\big(a_0|w_0|^2, |w_0|^2\big), \quad \left(\frac{a_0 w_0 \bar{w}_1 + a_1 w_1 \bar{w}_0}{2}, \frac{w_0 \bar{w}_1 + w_1 \bar{w}_0}{2}\right), \quad \big(a_1|w_1|^2, |w_1|^2\big). \tag{15}$$

Proof. Since $d^{-1} = \bar{d}/|d|^2$, we have

$$\xi(t) = qd^{-1} = \frac{q\bar{d}}{|d|^2} \tag{16}$$

$$= \frac{a_0|w_0|^2(1-t)^2 + (a_0 w_0 \bar{w}_1 + a_1 w_1 \bar{w}_0)(1-t)t + a_1|w_1|^2 t^2}{|w_0|^2(1-t)^2 + (w_0 \bar{w}_1 + w_1 \bar{w}_0)(1-t)t + |w_1|^2 t^2}, \tag{17}$$

i.e., $\xi(t)$ is a rational quadratic Bézier curve with three control points and real weights as in (15).

Note that we can change the weights w_0, w_1 to $1, w_1 w_0^{-1}$, then the arc $\xi(t)$ will remain the same. Moreover, if we change the parameter t to $\rho t/(1 - t + \rho t)$ ($\rho \in \mathbb{R}$) and weights w_0, w_1 to $w_0, w_1/\rho$, then the arc will also remain the same. Therefore, we can always assume that the weights are normalized, i.e., $w_0' = 1$, $w_1' = w_1 w_0^{-1}|w_0||w_1|^{-1} = w_1 \bar{w}_0/(|w_0||w_1|)$. Since $|w_1'| = 1$, we have $w_1' =$

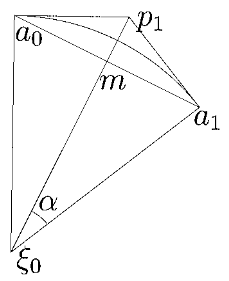

Figure 2. A center of an arc.

$(\cos\alpha, (\sin\alpha)u) = e^{\alpha u}$ for some α and $u \in \mathbb{R}^3$, $\langle u, u\rangle = 1$. Therefore, $(w_0'\bar{w}_1' + w_1'\bar{w}_0')/2 = \cos\alpha$. Hence,

$$\xi(t) = \frac{a_0(1-t)^2 + (a_0\bar{w}_1' + a_1 w_1')(1-t)t + a_1 t^2}{(1-t)^2 + 2(1-t)t\cos\alpha + t^2}, \tag{18}$$

i.e., $\xi(t)$ is a circle with three control points

$$P_0 = (a_0, 1), \quad P_1 = \big((a_0\bar{w}_1' + a_1 w_1')/2, \cos\alpha\big), \quad P_2 = (a_1, 1)$$

(see [3], Chapter 14).

For computation of the center ξ_0, note that three points $\xi_0, m = (a_0 + a_1)/2$ and $p_1 = (a_0\bar{w}_1' + a_1 w_1')/2\cos\alpha)$ are collinear. An elementary observation shows that $|p_1 m|/|m\xi_0| = \sin^2\alpha/\cos^2\alpha$ (see Fig. 2). Therefore, we obtain $m = (\sin^2\alpha)\xi_0 + (\cos^2\alpha)p_1$. Hence, $\xi_0 = (a_0 + a_1 - 2\cos^2\alpha p_1)/(2\sin^2\alpha)$. Since $\cos\alpha = (w_1 w_0^{-1} + \bar{w}_0^{-1}\bar{w}_1)|w_0||w_1|^{-1}/2 = (w_0\bar{w}_1 + w_1\bar{w}_0)/(2|w_0||w_1|)$, we prove the formula for the center. The formula for the radius follows from the sinus theorem.

Remark 3. If we take $w_0 = a_1 - b'$ and $w_1 = b' - a_0'$, where $a_0' = a_1^{-1}a_0 a_1$, $b' = (a_1 - a_0)^{-1}b(a_1 - a_0')$, then $\xi(1/2) = b$. Indeed, $\xi(1/2) = (a_0(a_1 - b') + a_1(b' - a_0'))(a_1 - b' + b' - a_0')^{-1} = (a_1 - a_0)b'(a_1 - a_0')^{-1} = b$. So, for the construction of the circular arc with three given points a_0, b, a_1, we can take weights w_0, w_1 as above.

The following proposition shows how to use pure imaginary quaternions for modelling a circle in $\mathbb{R}^3$.

PROPOSITION 4. *Let* $a_0 = (0, q_0), a_1 = (0, q_1) \in Im(\mathbb{H}) = \mathbb{R}^3$, *and* $w_1\bar{w}_0 = (r, p)$ *be such that* $\langle q_0, p\rangle = \langle q_1, p\rangle$. *Then* $\xi(t) \in Im(\mathbb{H}) = \mathbb{R}^3$ *and* $\xi(t)$ *belongs to the plane* L *in* $Im(\mathbb{H}) = \mathbb{R}^3$ *which is orthogonal to the vector* p *for any* t. *Moreover, if* $a_1 = w_1 w_0^{-1} a_0 w_0 w_1^{-1}$, *then the arc* $\xi(t)$ *is the trace of the rotation around the vector* p *from the point* q_0 *to the point* q_1 .

Proof. By Proposition 3 $\xi(t)$ has a representation as a Bézier curve with three control points a_0, p_1, a_1. Since $p_1 = (a_0 w_0 \bar{w}_1 + a_1 w_1 \bar{w}_0)/(w_0 \bar{w}_1 + w_1 \bar{w}_0) = (\langle q_1, p\rangle - \langle q_0, p\rangle, r(q_0 + q_1) + (q_0 - q_1) \times p)/(w_0 \bar{w}_1 + w_1 \bar{w}_0) \in Im(\mathbb{H})$ for any t, we have $\xi(t) \in Im(\mathbb{H})$. Moreover, $\xi(t)$ belongs to the plane L which passes through three points a_0, p_1, a_1. In this plane, we have two vectors $Im(p_1 - a_0)$ and $Im(p_1 - a_1)$. Since

$$(p_1 - a_0) = (a_1 - a_0) w_1 \bar{w}_0/(w_0 \bar{w}_1 + w_1 \bar{w}_0) = \big(0, r(q_1 - q_0) + (q_1 - q_0) \times p\big),$$

we see that $\langle Im(p_1 - a_0), p\rangle = 0$. In a similar way, we show that $\langle Im(p_1 - a_1), p\rangle = 0$. Hence, the plane L is orthogonal to the vector p.

If we take $a_1 = w_1 w_0^{-1} a_0 w_0 w_1^{-1}$, then

$$\xi(t) = (1 - t + w_1 w_0^{-1} t) a_0 w_0 \big((1 - t + w_1 w_0^{-1} t) w_0\big)^{-1} \tag{19}$$

$$= (1 - t + w_1 w_0^{-1} t) a_0 (1 - t + w_1 w_0^{-1} t)^{-1}. \tag{20}$$

Let $x = |x| x_n = |x|(\cos\alpha, (\sin\alpha) p)$ be given. It is well known that the map $Im(\mathbb{H}) = \mathbb{R}^3 \to Im(\mathbb{H}) = \mathbb{R}^3,\ q \to x q x^{-1} = x_n q \bar{x}_n$, is a rotation of the vector $Im(q)$ around the vector p by the angle 2α (see, for example, [4]). The formula (20) shows that $\xi(t)$ is a rotation around the vector $Im(w_1 w_0^{-1}) = p$. Since $\xi(t), t \in [0, 1]$ is an arc of the circle with end points a_0, a_1, we see that $\xi(t)$ is the trace of the rotation around the vector from the point $Im(a_0) = q_0$ to the point $Im(a_1) = q_1$.

5. SPLINES

LEMMA 5. *Let $\xi(t) = q(t)(d(t))^{-1} \in \mathbb{H}$. Then we have*

$$\xi'(t) = q'(t)(d(t))^{-1} - q(t)\bar{d}(t) d'(t) \bar{d}(t)/|d(t)|^4, \tag{21}$$

where $f'(t)$ denotes the derivative of a function $f(t)$ at a point t.

Proof. It is easy to see that $(q_1(t) q_2(t))' = q_1'(t) q_2(t) + q_1(t) q_2'(t)$ and $(d^{-1}(t))' = (\bar{d}(t)/|d(t)|^2)' = (\bar{d}'|d|^2 - \bar{d}(d\bar{d})')/|d|^4 = -\bar{d} d' \bar{d}/|d|^4$. The combination of these two formulas gives (21).

Now it is easy to find the derivatives for $\xi(t)$ at the end points (see also [6]):

$$\xi'(0) = v_0 = (a_1 - a_0) w_1 {w_0}^{-1}, \tag{22}$$

$$\xi'(1) = v_1 = (a_1 - a_0) w_0 {w_1}^{-1}. \tag{23}$$

Remark 4. For the construction of a circle arc by two points a_0, a_1 and initial tangent vector v_0 pointing into the arc, we can use formula (22) for computing the weight $w_1 = (a_1 - a_0)^{-1} v_0$ (here we assume that $w_0 = 1$). This representation of a circle by two end points and an initial tangent is more intuitive than that be the weight w_1.

Now assume that we have three points $c_0 = (a_0 w_0, w_0)$, $c_1 = (a_1 w_1, w_1), c_2 = (a_2 w_2, w_2)$ and two arcs $\xi_1(c_0, c_1)(t), \xi_2(c_1, c_2)(t)$. We are going to find a G^1 rational (tangent continuity) spline, i.e.,

$$h\xi_1'(1) = \xi_2'(0), \quad \text{where } h > 0. \tag{24}$$

So, if we take any c_0, c_1 and a_2, condition (24) gives $w_2 = h(a_2 - a_1)^{-1}(a_1 - a_0)w_0$, and we obtain a G^1 spline.

Definition. We say that the sequence $c_k = (a_k w_k, w_k) \in \mathbb{C}^2, k = 0, \ldots, n,$ is G^1 compatible if

$$w_{k+2} = h_{k+2}(a_{k+2} - a_{k+1})^{-1}(a_{k+1} - a_k)w_k, \quad 0 \leqslant k \leqslant n-2, \tag{25}$$

for some $h_i > 0$.

A G^1 compatible sequence defines the G^1 curve

$$C = \bigcup_{k=0}^{n-1} \xi(c_k c_{k+1})(t).$$

It is interesting to note that the curvature of this curve is the constant $1/R_k$ on every piece $\xi(c_k c_{k+1})(t)$, where R_k is the radius of the circle arc $\xi(c_k c_{k+1})(t)$. Therefore, the spline C is only a G^1 (not G^2) curve or a G^∞ if C is a circle.

There are a few examples of splines which are on one circle.

Example 1. If $c_0 = (a_0, 1), c_1 = (a_1 w_1, w_1), c_2 = (a_0 w_1^2, w_1^2)$, then $\xi(c_0 c_1)(t) \cup \xi(c_1 c_2)(t)$, $0 \leqslant t \leqslant 1$, is a full circle.

Example 2. If $c_0 = (a_0, 1)$, $c_1 = (a_1 w_1, w_1)$, $c_2 = (-a_1 w_1, -w_1)$, then $\xi(c_0 c_1)(t) \cup \xi(c_0 c_2)(t)$, $0 \leqslant t \leqslant 1$, is a full circle.

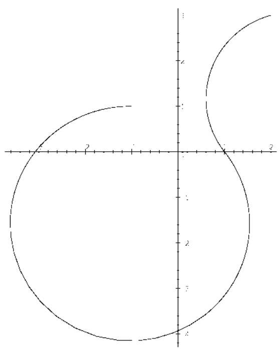

Figure 3. The spline of two circular arcs $\xi_1(t)$, $\xi_2(t)$.

Example 3. If $c_0 = (a_0, 1), c_1 = (a_1 i, i), c_2 = (-a_1 i, -i) \in \mathbb{C}^2$, then $\xi(c_0 c_1)(t) \cup \xi(c_0 c_2)(t),\ 0 \leqslant t \leqslant 1$, is a full circle in a plane with a center $(a_0 + a_1)/2$.

6. SUBDIVISION OF A CIRCULAR ARC

In this section, we describe subdivision of a circular arc. This does not gives nonstationary subdivision schemes which reproduce a circle as in the sense of book [7]. A circle reproducing subdivision scheme (in [7]) is a circle approximative scheme which uses linear formulas for computation of new control points by old points. Below, we present subdivision of a circular arc which is not linear and not approximative. For one step of this subdivision we use two control points and its weights. The computation of a new point and new weight is not linear and involves calculation of two square roots. While there is the advantage that computation of a new point involves only two points with weights, the disadvantage is that the coordinates of new points have to be calculated by a computation of square roots.

PROPOSITION 6. *Let $c_0 = (a_0 w_0, w_0)$ and $c_1 = (a_1 w_1, w_1) \in \mathbb{C}^2 (or\, \mathbb{H}^2)$ be as above, and let $c_{1/2} = (a_{1/2} w_{1/2}, w_{1/2}) = f(c_0, c_1)$, where*

$$c_{1/2} = c_0|w_1| + c_1|w_0| = \big(a_0 w_0|w_1| + a_1 w_1|w_0|,\, w_0|w_1| + w_1|w_0|\big), \textit{i.e.,} \tag{26}$$

$$a_{1/2} = \big(a_0 w_0|w_1| + a_1 w_1|w_0|\big)\big(w_0|w_1| + w_1|w_0|\big)^{-1}, \tag{27}$$

$$w_{1/2} = w_0|w_1| + w_1|w_0|. \tag{28}$$

Then three arcs $\xi(c_0, c_{1/2})(t),\ \xi(c_{1/2}, c_1)(t)$, and $\xi(c_0, c_1)(t)$ are on the same circle. Moreover,

$$|a_{1/2} - a_0| = |a_1 - a_{1/2}|, \tag{29}$$

i.e., the point $a_{1/2}$ is the middle point of the arc $\xi(c_0, c_1)(t)$.

Proof. An easy computation shows that

$$\xi(c_0, c_1)\left(\frac{|w_0|}{|w_0| + |w_1|}\right) = a_{1/2},$$

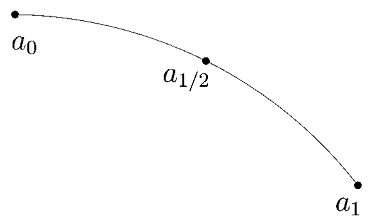

Figure 4. Uniform subdivision of an arc.

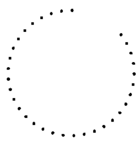

Figure 5. The subdivision of the arc $\xi(t)$, 33 points, $k = 5$.

$$\xi(c_0, c_1)\Big(\frac{|w_0|}{1 + |w_0| + |w_1|}\Big) = \xi(c_0, c_{1/2})\Big(\frac{1}{2}\Big),$$
$$\xi(c_0, c_1)\Big(\frac{1 + |w_0|}{1 + |w_0| + |w_1|}\Big) = \xi(c_{1/2}, c_1)\Big(\frac{1}{2}\Big).$$

Since two circle arcs $\xi(c_0, c_1)(t)$ and $\xi(c_0, c_{1/2})(t)$ have three common points $a_0, \xi(c_0, c_{1/2})(1/2), a_1$, we conclude that

$$\big\{\xi(c_0, c_{1/2})(t) \mid t \in [0, 1]\big\} \subset \big\{\xi(c_0, c_1)(t) \mid t \in [0, 1]\big\}.$$

In a similar way, we see that

$$\big\{\xi(c_{1/2}, c_1)(t) \mid t \in [0, 1]\big\} \subset \big\{\xi(c_0, c_1)(t) \mid t \in [0, 1]\big\}.$$

A straightforward computation shows that formula (29) also is true.

Now we can define a subdivision process. Let

$$c_{j/2^k} = f\big(c_{(j-1)/2^k}, c_{(j+1)/2^k}\big) = (a_{j/2^k} w_{j/2^k}, w_{j/2^k}) \tag{30}$$

by induction on k and for odd j. The points $a_{j/2^k}$ define a uniform subdivision of the arc $\xi(t)$.

If we take a G^1 compatible sequence c_k, $k = 0, \dots, n$, then the subdivision process gives points on a G^1 curve.

7. APPLICATIONS

A surface of revolution

Assume that a parametrization of a curve $C = \{q(u) \in \mathbb{R}^3\}$ is given. Then, using Proposition 4, we get the parametrization of a surface which is obtained as the rotation of the curve C around the axis p. Indeed, let $a_0 = (0, q(u))$, $w_0 = (1, 0)$, $w_1 = (0, p)$, $a_1 = w_1 a_0 w_1^{-1}$. Then we have

$$par(u, t) = \big(a_0 w_0 (1 - t) + a_1 w_1 (t)\big)\big(w_0 (1 - t) + w_1 (t)\big)^{-1}, \quad u, t \in [0, 1],$$

a parametrization of the surface of revolution.

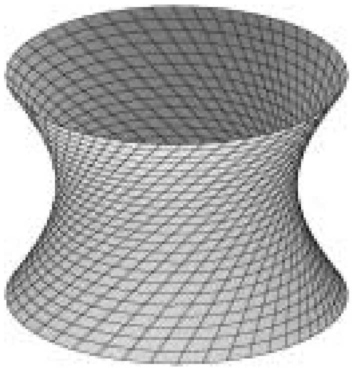

Figure 6. A hyperboloid.

For example, consider the parametrization of a segment of a line L through two points $q(u) = (1,0,0)(1-u) + (1,1,1)u$ and $p = (0,0,1)$. Then $par(u,t)$ is a parametrization of the hyperboloid which is the rotation of the line L segment around z axis (see Fig. 6).

A pipe surface

As an example, consider a part of a pipe surface smoothly blended with a cylinder C and a plane P. For the construction of this surface, we move around a rolling ball touching the cylinder and the plane. Let $L(s), s \in I$, denote a pencil of planes through the center line of the cylinder, and let $q_0(s) \in L(s) \cap C$ be the point where the rolling ball touches the cylinder line $L(s) \cap C$. Also, denote by $q_1(s)$ a point in the plane P where the rolling ball touches the line $L(s) \cap P$. We choose $a_0(s) = (0, q_0(s))$, $w_0 = 1$, $a_1(s) = (0, q_1(s))$ and by formula (22) we find $w_1(s) = (a_1(s) - a_0(s))^{-1}(0, v_0)$, where v_0 is the direction vector of the cylinder axis C. Now, the map

$$\xi(s,t) = (a_0(s)w_0(1-t) + a_1(s)w_1(s)t)(w_0(1-t) + w_1(s)t)^{-1}, \qquad t \in [0,1], s \in I, \tag{31}$$

gives a parametrization of a neck blending the cylinder with the plane (see Fig. 7). This is a part of the pipe surface.

For example, consider the cylinder C: $x^2 + y^2 = 1$ and the plane P: $z = ay$. Then the centers of rolling balls with radius r are on the plane P_1: $z = ay + r\sqrt{a^2+1}$ (since the distance between P and P_1 is r) and on the cylinder C_1: $x^2 + y^2 = (1+r)^2$. Let $[cs(s), sn(s)]$ be a parametrization of the circle $x^2 + y^2 = 1$, and let $L(s)$: $x \cdot sn(s) = y \cdot cs(s)$ be the plane through the center of the cylinder C. It is easy to see that $q_0(s) = [cs(s), sn(s), a \cdot sn(s) + r\sqrt{a^2+1}]$ is the point where the rolling ball touches the line $L(s) \cap C$ on the cylinder C. One can check that

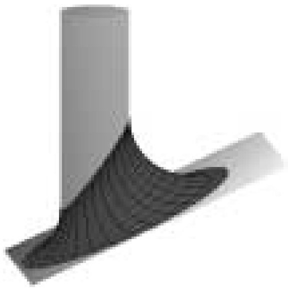

Figure 7. A part of the pipe surface smoothly connecting the cylinder and the plane.

$$q_1(s) = \Big[(r+1)cs(s), (r+1)sn(s), a(r+1)sn(s) + r\sqrt{a^2+1}\Big] + r[0, a, -1]/\sqrt{a^2+1}$$

is the point where the rolling ball touches the line $L(s) \cap P$. If we take $v_0 = [0, 0, -1]$ as the direction vector of cylinder axis and use parametrization (31), we obtain the blending of the plane and the cylinder see Fig. 7.

A circular surface

Another application of the previous construction can be a circular surface, i.e., a surface with one parameter family of circles. For example, consider two ellipses e_0: $x^2/a^2 + y^2/b^2 = 1$ and e_1: $x^2/c^2 + y^2/d^2 = 1$ in the plane P: $z = 0$ with the same center O. Let us denote by $q_0(s) = [a \cdot cs(s), b \cdot sn(s)]$, $q_1(s) = [c \cdot cs(s), d \cdot sn(s)]$ the parametrization of ellipses, where $cs(s)^2 + sn^2(s) = 1$. Assume that $a_0(s) = (0, (q_0(s), 0))$, $a_1(s) = (0, (q_1(s), 0))$, $w_0 = 1$, $v_0 = (0, 0, 0, 1)$ (here v_0 corresponds to the directional vector of lines on the elliptic cylinder). Then by formula (22) we compute $w_1(s) = (a_1 - a_0)^{-1} v_0$. Hence,

$$\xi(s,t) = \big(a_0(s)w_0(1-t) + a_1(s)w_1(s)t\big)\big(w_0(1-t) + w_1(s)t\big)^{-1} \tag{32}$$

is a parametrization of a circular surface. This surface intersects the plane P in two ellipses and all circles belong to planes perpendicular to the plane P (see Fig. 8). Note that this surface is not a canal surface. Then upper half of this surface may join smoothly (G^1-continuity) two elliptic cylinders (see Fig. 8).

A smooth “offset” of convex polyhedron

In the following example, we represent a smooth “offset” of convex polyhedron. By definition, this surface is the envelope surface of the rolling ball moving over the polyhedron. It is easy to see that the envelope consists of faces of the polyhedron moved

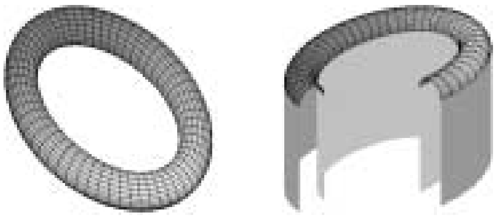

Figure 8. A circular surface and a part of it connecting two elliptic cylinders.

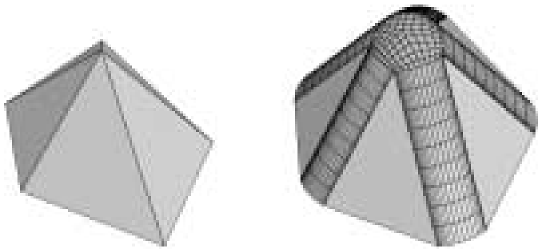

Figure 9. A polyhedron and a smooth "offset" of a polyhedron.

in its normal direction by a fixed distance. Instead of edges, we obtain a part of cylinders, and a vertex is replaced by an n-sided spherical patch. Using the construction of circular an arc (32) as above, we easily obtain a parametrization of parts for cylinders and n-angle in the sphere (see Fig. 9).

8. CONCLUSION AND FUTURE RESEARCH

We have represented a circular arc in the space by using end points and a vector perpendicular to the arc plane. Especially, this representation is useful for modelling surfaces with one parameter family of circles since in our construction of a circle, we do not need computation of the circle center.

There are possibilities for extensions and future research:

– Investigation of the geometry of higher degree Bézier curve (or surface) with complex (or quaternion) points and weights. Find an implicit equation for such curves (surfaces).
– An interesting extension concerns the Clifford algebra. For example, we can use control points and weights as elements of the Clifford algebra. Note that the quaternion algebra is isomorphic to the two dimensional Clifford algebra.

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REZIUMĖ

S. Zubė. Apskritimo konstravimas su kompleksiniais ir kvaternioniniais skaičiais

Darbe pateikta erdvinio apskritimo lanko dalies konstrukcija naudojant kvaterionus. Šios konstrukcijos privalumas prieš tradicinę Bezier apskritimo reprezentaciją yra tai, kad ji leidžia sumodeliuoti apskritimo lanką žinant tik jo pradžią, galą ir pradžios (arba galo) liestinės vektorių. Tai labai supaprastina apskritimų šeimos modeliavimą, nes nereikia rasti apskritimo centro ir lanko kampo. Darbe pateikti paviršių modeliavimo pavyzdžiai gauti naudojant minėtą konstrukciją. Taip pat aprašyti apskritiminiai splainai ir pristatytas jų padalinimo algoritmas.