Lectures in Pure and Applied Mathematics | 6

Suren Poghosyan | Mathias Rafler | Sylvie Roelly (Eds.)

Proceedings of the XI international conference Stochastic and Analytic Methods in Mathematical Physics

Potsdam University Press

11

The peak model for finite rank supersingular perturbations

Rytis Juršėnas[*]

Abstract. *We review the peak model for finite rank supersingular perturbations of a lower semibounded self-adjoint operator by comparing the main aspects with the A-model. The exposition utilies the techniques based on the notion of boundary triples.*

1 Introduction

Given a densely defined symmetric operator in a Hilbert space, there always exists a self-adjoint extension to a Hilbert space containing the initial one as a subspace. Adapting the present principle to the symmetric operator which is essentially self-adjoint, non-trivial extensions are constructed by extension-restriction procedure with respect to the triplet extensions in scales of Hilbert spaces of an initially given self-adjoint operator. Having found the Hilbert subspace in which the symmetric operator has non-trivial but finite defect numbers, one extends that subspace by a suitable finite dimensional linear space, and then considers triplet extensions restricted to the resultant space, which is equipped with an appropriate metric.

[*]Vilnius University, Institute of Theoretical Physics and Astronomy, Saulėtekio ave. 3, LT-10257 Vilnius, Lithuania; *rytis.jursenas@tfai.vu.lt*
The research is supported by the Research Council of Lithuania. The European Social Fund Grant No. 09.3.3-LMT-K-712-13-0025.

Depending on the inner structure of an additional linear space, and hence on the metric of a resultant space, one deals with the triplet extensions restricted to either Pontryagin or Hilbert space. Following [12], the Pontryagin space models are referred to as the B-models, while those which admit both indefinite and non-negative metrics are called A-models. The A- and B-models constitute the cascade models, since an additional finite-dimensional linear space in these models contains singular elements of different order of singularity, which therefore belong to different spaces from the scale. For the symmetric operator with defect numbers $(1,1)$ (in a subspace of an initial Hilbert space), the cascade models are developed in [12, 13, 16, 7]; see also the references therein.

Due to the indefiniteness of the metric in the cascade models, the so-called peak model was suggested in [18] as an alternative. The present model is purely Hilbert space model, but it has its own limitations, simply because the model does not apply to all operators (see the next paragraph for details).

In the present paper, we review the peak model for the restricted symmetric operator with defect numbers (d,d), $d\in\mathbb{N}$, which is developed in [14]. In parallel, we remark the key differences between the present model and the A-model. The results are presented using the techniques from the theory of boundary triples [8, 11, 10, 9].

2 The peak model versus A-model

In this section we construct non-trivial realisations of a symmetric operator that is essentially self-adjoint in the reference Hilbert space. The main results are the Krein-Naimark resolvent formula (11.1) and the computed Weyl function (11.3).

2.1 Triplet adjoint

As is well-known, non-trivial realisations of a symmetric operator $L_{\min}$ that is essentially self-adjoint in the reference Hilbert space $\mathfrak{H}_0$ are considered in an extended Hilbert space by means of the compressions of their resolvents. Thus, given a self-adjoint operator L in $\mathfrak{H}_0$, let $(\mathfrak{H}_n)_{n\in\mathbb{Z}}$ be the scale of Hilbert spaces [3, 1] associated with L. To simplify the present exposition, the operator L is lower semibounded. The scalar product in $\mathfrak{H}_n$ is defined by $\langle\cdot,\cdot\rangle_n := \langle\cdot, b_n(L)\cdot\rangle_0$, where $b_0(L):=I$ and $b_m(L):=(L-z_1)\cdots(L-z_m)$ and

$b_{-m}(L) := b_m(L)^{-1}$ for $m \in \mathbb{N}$. The real numbers $z_1, \dots, z_m$ from the resolvent set $\operatorname{res} L$ are fixed and referred to as the model parameters.

Let $L_{\min} \subseteq L$ be the symmetric restriction to $\mathfrak{H}_m$ with defect numbers (d,d), and the deficiency subspace spanned by the elements $\{G_\sigma(z) \in \mathfrak{H}_m \setminus \mathfrak{H}_{m+1}\}$, $z \in \operatorname{res} L$, with σ ranging over an index set $\mathscr{S}$ of cardinality $d \in \mathbb{N}$. One considers the triplet adjoint $L_{\max}$ of $L_{\min}$ for the Hilbert triple $\mathfrak{H}_m \subseteq \mathfrak{H}_0 \subseteq \mathfrak{H}_{-m}$. The operator $L_{\max}$ in $\mathfrak{H}_{-m}$ extends $L|_{\mathfrak{H}_{-m+2}}$ to the domain $\mathfrak{H}_{-m+2} \dotplus \mathfrak{N}_z(L_{\max})$ (direct sum), where the eigenspace $\mathfrak{N}_z(L_{\max})$ is the linear span of the singular elements $\{g_\sigma(z) := b_m(L) G_\sigma(z) \in \mathfrak{H}_{-m} \setminus \mathfrak{H}_{-m+1}\}$. These elements are also represented as the generalised vectors $g_\sigma(z) = (L-z)^{-1}\varphi_\sigma$ by means of linearly independent singular functionals $\{\varphi_\sigma \in \mathfrak{H}_{-m-2} \setminus \mathfrak{H}_{-m-1}\}$. The action of φ_σ on $\mathfrak{H}_{m+2}$ is realised via the duality pairing $\langle \varphi_\sigma, \cdot \rangle$ in a usual way [2, Eq. (1.17)]. Because $m \geq 1$, rank-d perturbations of L are called supersingular [17]; this is an allusion to the heuristic form $L + \sum_{\sigma,\sigma' \in \mathscr{S}} C_{\sigma\sigma'} \langle \varphi_{\sigma'}, \cdot \rangle \varphi_\sigma$ with some matrix $(C_{\sigma\sigma'})$ in $\mathbb{C}^d$. In what follows we also use the vector notation $\langle \varphi, \cdot \rangle := (\langle \varphi_\sigma, \cdot \rangle) \colon \mathfrak{H}_{m+2} \to \mathbb{C}^d$.

2.2 Intermediate space

To construct non-trivial realisations of $L_{\min}$, the space $\mathfrak{H}_{-m}$ in which $L_{\max}$ is defined turns out to be too large. Thus one defines the so-called intermediate space $\mathscr{H}$, in the sense that $\mathfrak{H}_m \subseteq \mathscr{H} \subseteq \mathfrak{H}_{-m}$, and considers the range restriction $A_{\max}$ to $\mathscr{H}$ of $L_{\max}$. As a linear space, $\mathscr{H}$ is the direct sum of $\mathfrak{H}_m$ and a md-dimensional linear space $\mathfrak{K} \subseteq \mathfrak{H}_{-m}$ such that $\mathfrak{K} \cap \mathfrak{H}_{m-1} = \{0\}$. Since $\mathfrak{K}$ is in bijective correspondence with $\mathbb{C}^{md}$, each element $k \in \mathfrak{K}$ is uniquely determined by the vector $d(k) \in \mathbb{C}^{md}$. Depending on the inner structure of $\mathfrak{K}$, the set $\mathscr{H}$ is made into either Hilbert or Pontryagin space by completing it with respect to the metric

$$\langle f + k, f' + k' \rangle_{\mathscr{H}} := \langle f, f' \rangle_m + \langle d(k), \mathscr{G} d(k') \rangle_{\mathbb{C}^{md}}$$

for $f, f' \in \mathfrak{H}_m$ and $k, k' \in \mathfrak{K}$, and some Hermitian matrix $\mathscr{G}$ in $\mathbb{C}^{md}$, referred to as the Gram matrix. For a suitable $\mathscr{G}$, the operator $A_{\max}$ is the adjoint in $\mathscr{H}$ of a densely defined, closed, symmetric, and simple operator $A_{\min}$; hence one applies to $A_{\min}$ a standard extension theory by means of $A_{\min} \subseteq A_\Theta \subseteq A_{\max}$, where a (closed) proper extension A_Θ is uniquely determined by a (closed) linear relation Θ in $\mathbb{C}^d$. Let $(\mathbb{C}^d, \Gamma_0, \Gamma_1)$ be an ordinary boundary triple (OBT) [11, Definition 7.11] for $A_{\max} = A^*_{\min}$, let γ and M be the corresponding γ-field and the Weyl function; then the Krein-Naimark resolvent formula for an extension A_Θ defined on $f \in \operatorname{dom} A_{\max}$ such that $(\Gamma_0 f, \Gamma_1 f) \in \Theta$ reads

$$(A_\Theta - z)^{-1} = (A_0 - z)^{-1} + \gamma(z)\big(\Theta - M(z)\big)^{-1}\gamma(\bar{z})^* \tag{11.1}$$

for $z \in \operatorname{res} A_\Theta \cap \operatorname{res} A_0$, where $A_0 := A_{\{0\}\times\mathbb{C}^d}$ is one of the two distinguished self-adjoint extensions of $A_{\min}$. From here one deduces the formula for the compression of the resolvent to $\mathfrak{H}_m$.

2.3 Gram matrix

In the cascade models, an Hermitian matrix $\mathscr{G}$ is initially chosen arbitrarily and the set $\mathfrak{K}$ is the linear span of the singular elements $h_\alpha := b_j(L)^{-1}\varphi_\sigma \in \mathfrak{H}_{-m-2+2j} \setminus \mathfrak{H}_{-m-1+2j}$, with $\alpha = (\sigma, j)$ ranging over $\mathscr{S} \times J$, $J := \{1, \dots, m\}$. However, the definition $A_{\min} := A_{\max}^*$ requires in addition that $\mathscr{G}$ be invertible. Moreover, to make $A_{\min}$ symmetric, the computation of the boundary form of $A_{\max}$ shows that $\mathscr{G}$ must satisfy a certain commutation relation. For example, when $d = 1$ and $z_1 = \ldots = z_m$, it must hold $\mathscr{G}\mathfrak{M} = \mathfrak{M}^*\mathscr{G}$ with a Hankel (anti-triangular) matrix $\mathfrak{M}$, i. e. the matrix with the entries $\mathfrak{M}_{jj'} := \delta_{jj'}z_1 + \delta_{j+1,j'}$ ($j \in J \setminus \{m\}$, $j' \in J$) and $\mathfrak{M}_{mj'} := \delta_{j'm}z_1$; for $m = 1$ one puts $\mathfrak{M} := z_1$. It follows in particular that, for $m \geq 2$, one cannot put $\mathscr{G} = \mathscr{G}_* := \big(\langle h_\alpha, h_{\alpha'}\rangle_{-m}\big)$, because $\langle h_{\sigma 1}, h_{\sigma 1}\rangle_{-m} > 0$ (this statement applies to $d \in \mathbb{N}$; see also [15]).

In contrast, in the peak model, the origin of $\mathscr{G}$ is clear. Namely, the Gram matrix $\mathscr{G}$ of the peak model is made of the entries $\mathscr{G}_{\alpha\alpha'} := \langle g_\alpha, g_{\alpha'}\rangle_{-m}$, where $g_\alpha := g_\sigma(z_j)$; hence it is Hermitian and positive definite provided that $z_j \neq z_{j'}$ for $j \neq j'$. The set $\mathfrak{K}$ is defined as the linear span of the singular elements of the same order of singularity, namely $\{g_\alpha \in \mathfrak{H}_{-m} \setminus \mathfrak{H}_{-m+1}\}$. It follows that each $k \in \mathfrak{K}$ is in bijective correspondence with $d(k) = \big(d_\alpha(k)\big) \in \mathbb{C}^{md}$ via

$$k = \sum_\alpha d_\alpha(k) g_\alpha, \qquad d_\alpha(k) = \sum_{\alpha'} \big[\mathscr{G}^{-1}\big]_{\alpha\alpha'} \langle g_{\alpha'}, k\rangle_{-m}.$$

In particular, using that

$$b_m(L)^{-1} = \sum_j b_j'(z_j)^{-1}(L - z_j)^{-1}, \qquad b_j'(\,\cdot\,) := \prod_{j' \in J \setminus \{j\}} (\,\cdot - z_{j'})$$

and putting $d_{\sigma j}(k) = c_\sigma b'_j(z_j)^{-1}$ for some $c = (c_\sigma) \in \mathbb{C}^d$, one deduces that the set $\mathfrak{K}_{\min} := \mathfrak{K} \cap \mathfrak{H}_{m-2} \setminus \mathfrak{H}_{m-1}$ is the linear span of $\{b_m(L)^{-1}\varphi_\sigma\}$, and is referred to as the minimal subset of $\mathfrak{K}$. An element $k \in \mathfrak{K}_{\min}$ is thus of the form $k = k_{\min}(c)$, where

$$k_{\min}(c) := \sum_\sigma c_\sigma b_m(L)^{-1}\varphi_\sigma = \sum_\alpha \left[\mathscr{G}^{-1}\mathscr{G}_b c\right]_\alpha g_\alpha.$$

The matrix $\mathscr{G}_b$ from $\mathbb{C}^d$ to $\mathbb{C}^{md}$ is formed by the entries

$$\left[\mathscr{G}_b\right]_{\alpha\sigma'} := \sum_{j'} \mathscr{G}_{\alpha,\sigma' j'} b'_{j'}(z_{j'})^{-1}$$

and has the trivial kernel.

2.4 Symmetric operator in intermediate space

The maximal operator $A_{\max}$ in the peak model is then the operator in the Hilbert space $\mathscr{H}$ which extends A_0 to the domain $\mathrm{dom} A_0 \dotplus \mathfrak{N}_z(A_{\max})$ for $z \in \mathrm{res} A_0 = \mathrm{res} L \setminus \{z_j \mid j \in J\}$, where the eigenspace of $A_{\max}$ coincides with that of $L_{\max}$ (but for $z \in \mathrm{res} A_0$). The minimal operator $A_{\min}$ is made symmetric iff $\mathscr{G}$ is diagonal in $j \in J$, in which case $A_{\max}$ is closed and equals $A_{\max} = A^*_{\min}$ (this is in contrast to the A-model, where $A_{\max}$ is automatically closed by construction, provided that the Gram matrix of the model is invertible):

$$\mathrm{dom} A_{\min} = \{f^\# + k \in \mathfrak{H}_{m+2} \dotplus \mathfrak{K} \mid \langle \varphi, f^\# \rangle = \mathscr{G}^*_b d(k)\},$$
$$\mathrm{dom} A_{\max} = \mathrm{dom} A_0 \dotplus \mathfrak{N}_z(A_{\max}) = \mathfrak{H}_{m+2} \dotplus \mathfrak{N}_z(L^*_{\min}) \dotplus \mathfrak{K}$$

and

$$A_{\max}\left(f^\# + G_z(c) + k\right) = A_0(f^\# + k) + zG_z(c) + k_{\min}(c), \tag{11.2}$$
$$G_z(c) := \sum_\sigma c_\sigma G_\sigma(z), \qquad c = (c_\sigma) \in \mathbb{C}^d, \qquad z \in \mathrm{res} A_0$$

where the self-adjoint operator A_0 on $\mathrm{dom} A_0 = \mathfrak{H}_{m+2} \dotplus \mathfrak{K}$ is defined by

$$A_0(f^\# + k) = Lf^\# + \sum_\alpha \left[Z_d d(k)\right]_\alpha g_\alpha.$$

For brevity, Z_d denotes the matrix direct sum of d diagonal matrices $\mathrm{diag}\{z_j; j \in J\}$.

On one hand, the diagonality of $\mathscr{G}$ significantly simplifies the computations, but on the other hand, the condition is not satisfied for some operators L with perturbations of class $\mathfrak{H}_{-m-2} \setminus \mathfrak{H}_{-m-1}$ with $m \geq 2$; see e. g. [15] for $m = 2$.

2.5 Weyl function

In the peak (resp. cascade) model, the Weyl function M is represented by the sum of a Nevanlinna function associated with $L_{\min}$ in $\mathfrak{H}_m$ and the Krein Q-function associated with the Gram matrix $\mathscr{G}$ (resp. the generalszed Nevanlinna function – due to the indefiniteness of $\mathscr{G}$ – associated with the multiplication operator in a reproducing kernel Pontryagin space [5, 4, 6]). More specifically, the γ-field and the Weyl function associated with the OBT

$$\Gamma_0(f^\# + G_z(c) + k) := c, \qquad \Gamma_1(f^\# + G_z(c) + k) := \langle \varphi, f^\# \rangle + R(z)c - \mathscr{G}_b^* d(k)$$

for $A_{\max}$ are given by

$$\gamma(z) = b_m(z)^{-1} b_m(L) G_z(\cdot), \qquad M(z) = R(z) + Q_{\mathscr{G}}(z) \tag{11.3}$$

for $z \in \operatorname{res} A_0$. Here R is the Weyl function associated with the OBT for $L^*_{\min}$, which is obtained from $\Gamma := (\Gamma_0, \Gamma_1)$ by restriction to $\operatorname{dom} L^*_{\min} = \mathfrak{H}_{m+2} \dotplus \mathfrak{N}_z(L^*_{\min})$. Note that the γ-field associated with this OBT for $L^*_{\min}$ is $G_z(\cdot)$. The Q-function associated with $\mathscr{G}$ is the matrix in $\mathbb{C}^d$ whose entries are defined by

$$\left[Q_{\mathscr{G}}(z)\right]_{\sigma\sigma'} := \sum_j \frac{\left[\mathscr{G}_b^*\right]_{\sigma,\sigma' j}}{(z_j - z) b'_j(z_j)} = \sum_j \frac{\mathscr{G}_{\sigma j,\sigma' j}}{(z_j - z) b'_j(z_j)^2}.$$

The second equality accounts for the condition that $\mathscr{G}$ is diagonal in $j \in J$; for $d = 1$, see also [18, Theorem 6.1].

2.6 Renormalised Weyl function

Ignoring formally that $\mathscr{G}$ is diagonal in j for $m \geq 2$, one can perform a kind of renormalisation of $Q_{\mathscr{G}}(z)$. For this purpose, put $z_j = z_1 - \delta_{j-1}$, $\delta_{j-1} \neq 0$, $j \in J \setminus \{1\}$, in the first formula of $Q_{\mathscr{G}}(z)$, take the limits $\delta_j \to \delta_{j-1}$, as well as $\delta_1 \to 0$, and deduce by induction that the matrix $Q_{\mathscr{G}}(z)$ is "renormalised" to the matrix $Q_*(z)$ whose entries are given by

$$\big[Q_*(z)\big]_{\sigma\sigma'} := -\sum_j \frac{\big[\mathscr{G}_*\big]_{\sigma m,\sigma' j}}{(z-z_1)^{m-j+1}}.$$

An interesting observation is that the corresponding Weyl function M, denoted now by M_*, is, up to a constant, the Weyl function M_{A} of $A_{\max}$ in the A-model with model parameters $z_j = z_1$, provided that the entries at the m-th row of the Gram matrix of the A-model satisfy $\mathscr{G}_{\sigma m,\sigma' j} = \big[\mathscr{G}_*\big]_{\sigma m,\sigma' j}$. In this case, with a suitable choice of the OBT for $A_{\max}$ in the A-model, one has

$$M_*(z) = R_*(z_1) + M_{\mathrm{A}}(z)$$

for $z \in \operatorname{res} L \setminus \{z_1\}$. In the above formula R_* is obtained from R by simply replacing all $\{z_j\}$ in $b_m(L)$ by z_1; that is, the entries

$$\big[R_*(z) - R_*(z_1)\big]_{\sigma\sigma'} = (z-z_1)\,\langle \varphi_\sigma, b_m(L)^{-1}(L-z)^{-1}(L-z_1)^{-1}\varphi_{\sigma'}\rangle$$

for $z \in \operatorname{res} L$, constitute the matrix valued Q-function which is associated with $L_{\min}$ in $\mathfrak{H}_m$, where now $b_m(L) := (L-z_1)^m$.

3 Transformation preserving the Weyl function

According to [19], if Q-functions of two densely defined, closed, symmetric, and simple operators in (possibly) distinct Hilbert spaces coincide, then the operators are unitarily equivalent. In this paragraph we extend the latter statement to a not necessarily unitary transformation, which becomes unitary, however, in the special case.

Let $P_{\mathscr{H}}$ be a bounded operator from a Hilbert space $\mathfrak{H}_{-m}$ to a Hilbert space $\mathscr{H}$; let $P_{\mathscr{H}}^*$ be its adjoint, considered as a bounded operator from $\mathscr{H}$ to $\mathfrak{H}_{-m}$. Let $\Omega := P_{\mathscr{H}} b_m(L)^{1/2}$ be a bounded operator from $\mathfrak{H}_0$ to $\mathscr{H}$; then the operator $\Omega^* = b_m(L)^{-1/2} P_{\mathscr{H}}^*$, considered as a bounded operator from $\mathscr{H}$ to $\mathfrak{H}_0$, is the adjoint of Ω. Define also a bounded, non-negative, self-adjoint operator in $\mathscr{H}$ by $\iota := \Omega\Omega^* = P_{\mathscr{H}} P_{\mathscr{H}}^*$.

Let A_Θ be a (closed) proper extension of the symmetric operator $A_{\min}$ in $\mathscr{H}$ as described above, and define the operator $\boldsymbol{A}_\Theta := \Omega^* A_\Theta \Omega$ in $\mathfrak{H}_0$ on its natural domain. A direct computation shows that the adjoint in $\mathfrak{H}_0$ is the operator $\boldsymbol{A}_{\Theta^*}$. Let also $\boldsymbol{A}_{\min} := \Omega^* A_{\min} \Omega$, and similarly for $\boldsymbol{A}_{\max}$. Then $\boldsymbol{A}_\Theta$ is a proper extension of a densely defined, closed, symmetric, and simple operator $\boldsymbol{A}_{\min}$. The domain of $\boldsymbol{A}_\Theta$ can be described in terms of Θ as the set of $u \in \operatorname{dom}\boldsymbol{A}_{\max}$ such that $(\boldsymbol{\Gamma}_0 u, \boldsymbol{\Gamma}_1 u) \in \Theta$, where $\boldsymbol{\Gamma} := (\boldsymbol{\Gamma}_0, \boldsymbol{\Gamma}_1)\colon \operatorname{dom}\boldsymbol{A}_{\max} \to \mathbb{C}^d \times \mathbb{C}^d$ is defined according to $\boldsymbol{\Gamma} = \Gamma\Omega$. Because $\Omega \operatorname{dom}\boldsymbol{A}_{\max} \subseteq \operatorname{dom}A_{\max}$ the operator $\boldsymbol{\Gamma}$ is not surjective, in general, so the present parametrisation of $\operatorname{dom}\boldsymbol{A}_\Theta$ applies to not all Θ, and the triple $\boldsymbol{\Pi} := (\mathbb{C}^d, \boldsymbol{\Gamma}_0, \boldsymbol{\Gamma}_1)$ is only an isometric boundary triple [8, Definition 1.8] for $\boldsymbol{A}_{\max}$. To make $\boldsymbol{\Pi}$ an OBT, we assume that $P_{\mathscr{H}}$ leaves $\operatorname{dom}A_{\max}$ invariant, because in this case $\Omega \operatorname{dom}\boldsymbol{A}_{\max} = \operatorname{dom}A_{\max}$. Then the following result holds.

Theorem 11.1 Let $\boldsymbol{M}$ be the Weyl function of $\boldsymbol{A}_{\min}$ corresponding to the OBT $\boldsymbol{\Pi}$ for $\boldsymbol{A}_{\max}$. Then $\boldsymbol{M}(z) = M(z)$, $z \in \operatorname{res}A_0$, iff

$$(\forall c \in \mathbb{C}^d)\,(\forall z \in \Sigma_\iota)\ \left[(A_0 - z)^{-1} - (\iota A_0 - z)^{-1}\iota\right] k_{\min}(c) \\ - (\iota A_0 - z)^{-1}(\iota - I) z G_z(c) \in \operatorname{dom}A_{\min}.$$

Here $\Sigma_\iota := \operatorname{res}A_0 \cap \operatorname{res}(\iota A_0)$. For $\iota = I$, one recovers that $P_{\mathscr{H}}$ (and hence Ω) is unitary, which is the case considered in [19, Theorem 2.2].

Proof. First, observe that $\mathfrak{N}_z(\iota A_{\max}) = H_z(\mathbb{C}^d)$, $z \in \Sigma_\iota$, where

$$H_z(c) := \left[I - z(\iota A_0 - z)^{-1}(\iota - I)\right] G_z(c) - (\iota A_0 - z)^{-1}\iota k_{\min}(c).$$

Indeed, since $f \in \mathfrak{N}_z(\iota A_{\max})$ belongs to $\operatorname{dom}A_{\max}$, it follows from (11.2) that

$$0 = \left(\iota A_0 - z\right)(f^\# + k) + (\iota - I) z G_z(c) + \iota k_{\min}(c).$$

By using $(L - z)G_z(c) = k_{\min}(c)$ the assertion follows.

Second, the graph of the γ-field $\boldsymbol{\gamma}$ associated with an OBT $\boldsymbol{\Pi}$ consists of the pairs $(c, u_z) \in \mathbb{C}^d \times \mathfrak{N}_z(\boldsymbol{A}_{\max})$ such that $\Omega u_z = H_z(c)$. Indeed, by definition, $\boldsymbol{\gamma}$ contains (c, u_z) such that $\boldsymbol{\Gamma}_0 u_z = c$. Since $\Omega \mathfrak{N}_z(\boldsymbol{A}_{\max}) \subseteq \mathfrak{N}_z(\iota A_{\max})$, the assertion follows by using the first claim. To verify that $\boldsymbol{\gamma}$ is the graph, let us compute its multivalued part; it is the set of $u_z \in \mathfrak{N}_z(\boldsymbol{A}_{\max}) \cap \ker\Omega = \ker(\boldsymbol{\Gamma}_0 \,|\, \mathfrak{N}_z(\boldsymbol{A}_{\max})) = \{0\}$.

Third, the Weyl function $\boldsymbol{M}$ associated with $\boldsymbol{\Pi}$ is given by $\boldsymbol{M}(z) = \Gamma_1 H_z(\cdot)$ (on $\mathbb{C}^d$) for $z \in \Sigma_\iota$, which is seen from the second claim. Using in addition that

$$Q_{\mathscr{G}}(z)c = -\Gamma_1\left[(A_0 - z)^{-1} k_{\min}(c)\right]$$

one gets that

$$\boldsymbol{M}(z) = M(z) + \Delta(z)$$

with

$$\Delta(z)c := \Gamma_1\{[(A_0 - z)^{-1} - (\iota A_0 - z)^{-1}\iota]k_{\min}(c) - (\iota A_0 - z)^{-1}(\iota - I)zG_z(c)\}.$$

Thus $\boldsymbol{M}(z) = M(z)$ iff $\Delta(z)$ vanishes; in this case the equality for the analytic Weyl functions extends to the domain of analyticity of $M(z)$, namely, $\operatorname{res} A_0$.

Finally, $(\forall c)$ $\Delta(z)c = 0$ iff the term in $\{\,\}$, which belongs to $\operatorname{dom} A_0$ by construction, also belongs to $\ker\Gamma_1$, i. e. iff it belongs to $\operatorname{dom} A_0 \cap \ker\Gamma_1 = \operatorname{dom} A_{\min}$. □

To this end we remark that a similar theorem can be formulated in the A-model as well, but now the situation is more delicate, because M in the A-model might belong to the class of generalised Nevanlinna families with a finite number κ of negative squares, while $\boldsymbol{M}$ associated with the OBT for the Hilbert space (i. e. $\mathfrak{H}_0$) adjoint of a symmetric operator belongs to the class of Nevanlinna families, i. e. $\kappa = 0$. Thus a different meaning has to be given to the adjoint of $P_{\mathscr{H}}$ (and hence Ω). The details will be presented elsewhere.

Bibliography

[1] Albeverio, S., Kurasov, P.: *Rank one perturbations of not semibounded operators*, Integr. Equ. Oper. Theory **27**, 379–400 (1997).

[2] Albeverio, S, Kurasov, P.: *Singular perturbations of differential operators*, London Mathematical Society Lecture Note Series 271. Cambridge University Press, UK (2000).

[3] Albeverio, S, Kuzhel, S., Nizhnik, L.: *Singularly perturbed self-adjoint operators in scales of Hilbert spaces*, Ukr. J. Math. **59**(6), 787–810 (2007).

[4] Behrndt, J., Hassi, S., de Snoo, H.: *Boundary relations, unitary colligations, and functional models*, Compl. Anal. Oper. Theory **3**(1), 57–98 (2009).

[5] Behrndt, J., Hassi, S., de Snoo, H.: *A realization theorem for generalized Nevanlinna families*, Oper. Matrices **5**(4), 679–706 (2011).

[6] Derkach, V.: *Abstract interpolation problem in Nevanlinna classes*, Modern Analysis and Applications **190**, 197–236 (2009).

[7] Derkach, V., Hassi, S., de Snoo, H.: *Singular perturbations of self-adjoint operators*, Math. Phys. Anal. Geom. **6**(4), 349–384 (2003).

[8] Derkach, V., Hassi, S., Malamud, M.: *Generalized boundary triples, Weyl functions and inverse problems*, *arXiv:1706.07948v1* (2017).

[9] Derkach, V., Hassi, S., Malamud, M., de Snoo, H.: *Boundary relations and their Weyl families*, Trans. Amer. Math. Soc. **358**(12), 5351–5400 (2006).

[10] Derkach, V., Hassi, S., Malamud, M., de Snoo, H.: *Boundary relations and generalized resolvents of symmetric operators*, Russ. J. Math. Phys. **16(1)**, 17–60 (2009).

[11] Derkach, V., Hassi, S., Malamud, M., de Snoo, H.: *Boundary triplets and Weyl functions. Recent developments*, In Seppo Hassi, Hendrik S. V. de Snoo, and Franciszek Hugon Szafraniec, editors, *Operator Methods for Boundary Value Problems, London Math. Soc. Lecture Note Series*, volume 404, chapter 7, pages 161–220. Cambridge University Press, UK (2012).

[12] Dijksma, A., Kurasov, P., Shondin, Yu.: *High order singular rank one perturbations of a positive operator*, Integr. Equ. Oper. Theory **53**, 209–245 (2005).

[13] Dijksma, A., Langer, H., Shondin, Yu.: *Rank one perturbations at infinite coupling in Pontryagin spaces*, J. Func. Anal. **209**(1), 206–246 (2004).

[14] Juršėnas, R.: *The peak model for the triplet extensions and their transformations to the reference Hilbert space in the case of finite defect numbers*, *arXiv:1810.07416* (2018).

[15] Juršėnas, R.: *On some extensions of the A-model*, *arXiv:1909.07078* (2019).

[16] Kurasov, P.: *$\mathfrak{H}_{-n}$-perturbations of self-adjoint operators and Krein's resolvent formula*, Integr. Equ. Oper. Theory **45**(4), 437–460 (2003).

[17] Kurasov, P.: *Singular and supersingular perturbations: Hilbert space methods*, Contemp. Math. **340**, 185–216 (2003).

[18] Kurasov, P.: *Triplet extensions I: Semibounded operators in the scale of Hilbert spaces*, J. Anal. Math. **107**(1), 252–286 (2009).

[19] Langer, H., Textorius, B.: *On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space*, Pac. J. Math. **72**(1), 135–165 (1977).