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Approximations for sums of three-valued 1-dependent symmetric random variables

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Abstract. The sum of symmetric three-point 1-dependent nonidentically distributed random variables is approximated by a compound Poisson distribution. The accuracy of approximation is estimated in the local and total variation norms. For distributions uniformly bounded from zero, the accuracy of approximation is of the order $O(n^{-1})$. In the general case of triangular arrays of identically distributed summands, the accuracy is at least of the order $O(n^{-1/2})$. Nonuniform estimates are obtained for distribution functions and probabilities. The characteristic function method is used.

Keywords: compound Poisson distribution, 1-dependent variables, total variation norm, local norm, nonuniform estimate.

1 Introduction

We consider the direct extension of 2-runs statistic to a symmetric case. Note that, due to its explicit structure, k-runs (and, especially, 2-runs) statistic is arguably the best investigated case of m-dependent rvs, see [2, 11, 20]. However, in the papers devoted to discrete approximations of weakly dependent random variables (rvs), it is typical to assume their nonnegativeness, for example, see [4, 14, 19, 20]. As far as we know, so far there was no attempt to apply compound Poisson approximation to the sums of 1-dependent rvs taking positive and negative values. Meanwhile, for independent rvs, it is well-known that symmetry can considerably improve the accuracy of approximation. Our goal is to demonstrate that similar improvement is also possible for 1-dependent rvs.

The 2-runs statistic can be expressed as a sum $S = \xi_1\xi_2 + \xi_2\xi_3 + \cdots + \xi_n\xi_{n+1}$, where all ξ_j are independent Bernoulli variables. In this paper, we retain a similar structure, replacing Bernoulli variables by symmetric three-point rvs. We use the difference of two independent Poisson variables with the same mean for approximation. Obviously, the corresponding rv is infinitely divisible and is a special case of compound Poisson (CP) rvs. Such a rv naturally occurs in various situations and its probabilities can be expressed as modified Bessel functions of the first kind, see [9, p. 198].

The 2-runs statistic can be associated with an electric circuit of n-switches having two positions (on and off), where current flows only if both consequent switches are on. Similarly, we can relate our scheme to switches with three positions. For example, imagine the game in which each player chooses between two options (say red and blue) or refuses to play. If the player chooses the same color as previous one, casino pays some amount to the player. If the player chooses different color from the previous player, then the same amount is paid by the player to the casino. If the player refuses to play or the previous player has refused to play, then no amount is paid by either side. The first choice is done by the casino. Then S is the aggregate amount of money paid by casino to the team of n players.

Next, we introduce some preliminary notation. Let $M_{\mathbb{Z}}$ be a set of finite signed measures concentrated on the set of all integers $\mathbb{Z}$. The Fourier transform of $M \in M_{\mathbb{Z}}$ is denoted by

$$\widehat{M}(t) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tk} M\{k\}, \quad t \in \mathbb{R}.$$

Characteristic functions and Fourier transforms are denoted by small Greek letters or the same capital letters as their measures with additional hats, for example, $\widehat{F}_n(t)$ denotes the characteristic function of F_n.

We define $M((-\infty, x]) := \sum_{j=-\infty}^{x} M\{j\}$. Let I_a denote the distribution concentrated at a point $a \in \mathbb{R}$, with $I \equiv I_0$. Then $\widehat{I}_a(t) = \mathrm{e}^{ita}$ and $\widehat{I}(t) = 1$. All products of measures are defined in the convolution sense, that is, for all finite signed measures F, G defined on the σ-field $\mathcal{B}$ of one-dimensional Borel subsets and Borel set X

$$FG\{X\} = \int_{\mathbb{R}} F\{X - x\}\, G\{\mathrm{d}x\}.$$

If $N, M \in M_{\mathbb{Z}}$, for a set $A \subseteq \mathbb{Z}$, $NM\{A\} = \sum_{k=-\infty}^{\infty} N\{A-k\}M\{k\}$ and $M^0 = I$. The exponential of M is given by

$$\mathrm{e}^M = \exp\{M\} := \sum_{k=0}^{\infty} \frac{1}{k!} M^k.$$

The CP distribution with compounding distribution F is defined as

$$\exp\bigl\{\lambda(F - I)\bigr\} = \sum_{m=0}^{\infty} \frac{\lambda^m (F-I)^m}{m!}, \quad \lambda > 0.$$

This is a generalization of the Poisson law $\exp\{\lambda(I_1 - I)\}$, $\lambda > 0$, see [3, p. 4].

In this paper, the accuracy of approximation is measured in the local, uniform (Kolmogorov) and total-variation norms defined by

$$\|M\|_\infty := \sup_{k\in\mathbb{Z}} \bigl|M\{k\}\bigr|, \qquad |M|_{\mathrm{K}} := \sup_{x\in\mathbb{R}} \bigl|M\bigl\{(-\infty, x]\bigr\}\bigr|, \qquad \|M\| := \sum_{j=-\infty}^{\infty} \bigl|M\{j\}\bigr|,$$

respectively. In the proofs, we apply the following well-known relations:

$$\widehat{MN}(t) = \widehat{M}(t)\widehat{N}(t), \qquad \|MN\| \leqslant \|M\|\|N\|, \qquad |MN|_{\mathrm{K}} \leqslant \|M\||N|_{\mathrm{K}},$$
$$\|MN\|_\infty \leqslant \|M\|\|N\|_\infty, \qquad \bigl|\widehat{M}(t)\bigr| \leqslant \|M\|, \qquad \widehat{\exp\{M\}}(t) = \exp\bigl\{\widehat{M}(t)\bigr\}.$$

The next section will provide the already known results related to the approximations of m-dependent random variables. After that, the obtained results are presented, followed by auxiliary results and proofs.

To measure similarity of two rvs we need a notation for mixed centered moments. Let $\widehat{\mathbf{E}}Y_1 = \mathbf{E}\,Y_1$. Then $\widehat{\mathbf{E}}(Y_1, Y_2, \dots, Y_k)$ is defined recursively by

$$\widehat{\mathbf{E}}(Y_1, Y_2, \dots, Y_k) = \mathbf{E}\,Y_1Y_2\cdots Y_k - \sum_{j=1}^{k-1} \widehat{\mathbf{E}}(Y_1, \dots, Y_j)\,\mathbf{E}\,Y_{j+1}\cdots Y_k.$$

This notation was introduced by Statulevičius [16]; see also [7] and the reference therein.

We denote by C all positive absolute constants. The letter θ stands for any complex number satisfying $|\theta| \leqslant 1$. The values of C and θ can vary from line to line or even within the same line. Sometimes we supply constants with indices.

2 Known results

CP and signed CP approximations are frequently applied in insurance models and in limit theorems because the accuracy of such approximations can be of better order than the limit normal distribution (see [5, 8] and the reference therein). For example, in his seminal paper Prokhorov showed that Poisson approximation to binomial distribution can be more accurate than the normal one (see [13]). It is also well-known that symmetry can significantly improve the accuracy of CP approximation. Thus, two of the most general results for sums of independent symmetric rvs state that

$$\sup_{F \in \mathcal{F}_s} \bigl|F^n - \exp\bigl\{n(F - I)\bigr\}\bigr|_K \leqslant Cn^{-1/2}, \tag{1}$$
$$\sup_{F \in \mathcal{F}_+} \bigl|F^n - \exp\bigl\{n(F - I)\bigr\}\bigr|_K \leqslant Cn^{-1}. \tag{2}$$

Here the first supremum is taken over all symmetric distributions $\mathcal{F}_s$, and the second supremum is taken over all distributions with nonnegative characteristic functions $\mathcal{F}_+$. Estimates (1) and (2) were proved respectively by Zaitsev and Arak, see [1].

Additional moment assumptions are needed if one wants to replace the Kolmogorov norm by a stronger total variation norm in (2).

The sum of m-dependent rvs have been thoroughly approximated by a normal distribution (see, for example, [15]). It has properties similar to the well-known Berry–Essen theorem, i.e., *in the scheme of sequences*, the accuracy of approximation in the Kolmogorov norm is of the order $O(n^{-1/2})$. Note that a *scheme of sequences* refers to iid rvs $\xi_1, \xi_2, \dots, \xi_n$ having distribution F, where the characteristics of F do not depend on n.

In principle, research can be extended to the symmetric case. However, not much can be said about the accuracy of normal approximation if we consider the triangular arrays (when all parameters can depend on the number of summands n, i.e., for each n, we consider different sets of rvs $\xi_{1n}, \xi_{2n}, \dots, \xi_{nn}$) and if we want to use the total variation norm which is stronger than the Kolmogorov norm.

We are aware about just few results related to Poisson-type approximations to symmetric weakly dependent rvs. The sums of symmetric Markov-dependent rvs with three states are approximated by the CP distribution in [18]. A more general nonsymmetric case is analyzed in [10]. It was proved that, under the assumption on the smallness of transition probabilities, some analogue of (2) holds for the total variation metric. Notably the symmetrized Pólya–Aeppli distribution is used as CP approximation.

Our aim is to prove similar results for the triangular array of symmetric 1-dependent rvs. As we already mentioned in Introduction, we consider the extension of 2-runs statistic to a symmetric case. Our assumptions are formulated more precisely in the next section.

3 Setting

We assume that

- ξ_j $(j = 1, 2, \dots)$ are independent (not necessarily identically distributed) symmetric rvs with $\mathbf{P}(\xi_j = -1) = p_j$, $\mathbf{P}(\xi_j = 0) = 1 - 2p_j$, and $\mathbf{P}(\xi_j = 1) = p_j$.
- $X_j = \xi_j \xi_{j+1}$, $S_n = X_1 + \dots + X_n$.

All results are proved under assumption

$$p := \max p_j \leqslant \frac{1}{24}. \tag{3}$$

Observe that, in general, we consider triangular arrays, that is, p_j, $j = 1, 2, \dots$, can depend on n. Further on, we also use the following notations:

- $\sigma^2 = \mathbf{Var}\, S_n$,
- F_n is a distribution of S_n,
- $\varepsilon = \sum_{k=2}^{n-1} (p_{k-1} p_k p_{k+1} + (p_k p_{k+1})^2)$,
- $Y_j = \mathrm{e}^{\mathrm{i}tX_j} - 1,\ z = \mathrm{e}^{\mathrm{i}t} - 1,\ \bar{z} = \mathrm{e}^{-\mathrm{i}t} - 1$,
- $G = \prod_{k=1}^{n} G_k,\ G_k = \exp\{2p_k p_{k+1}(I_1 + I_{-1} - 2I)\}$.

Notice that $\widehat{G}_k(t) = \exp\{2p_k p_{k+1}(\mathrm{e}^{\mathrm{i}t} - 1)\} \exp\{2p_k p_{k+1}(\mathrm{e}^{-\mathrm{i}t} - 1)\}$, that is, G_k is the distribution of the difference of two independent Poisson rvs with the same mean $2p_k p_{k+1}$.

It is obvious that previously defined X_j are 1-dependent rvs because X_j depends only on X_{j-1} and X_{j+1} for all $j \geqslant 2$. We recall that a sequence of rvs Z_k, $k = 1, \dots$, is called m-dependent if, for $1 < s < t < \infty$, $t - s > m$, the σ-algebras generated by $Z_1, \dots, Z_s$ and by $Z_t, Z_{t+1}, \dots$ are independent.

Observe that the sum of m-dependent rvs can be easily reduced to the sum of 1-dependent rvs by grouping consecutive summands. The characteristic function method is used in the proofs. We apply Heinrich's adaptation of the characteristic function method for weakly dependent random variables (see [6, 7], also [3, Chap. 13]).

4 Results

The following theorem is the main result of this paper. It includes the general approximation result for F_n.

Theorem 1. *Let condition* (3) *be satisfied. Then, for all* $n = 2, 3 \dots,$

$$\|F_n - G\|_\infty \leqslant C\varepsilon \min(\sigma^{-5}, 1), \qquad \|F_n - G\| \leqslant C\varepsilon \min(\sigma^{-4}, 1). \tag{4}$$

Since $|F_n - G|_{\mathrm{K}} \leqslant \|F_n - G\|$, the accuracy in Kolmogorov norm is at least as in (4).

Corollary 1. *If* $0 < \partial \leqslant p_k \leqslant p \leqslant 1/24$ *for all* $k = 1, 2, \dots, n$ *and for some* $\bar{p}$, p, *then*

$$\|F_n - G\| \leqslant C \min\left(\frac{p^3}{n\partial^4}, np^3\right).$$

Corollary 2. *If* $p_k \equiv p > 0$ *for all* $k = 1, 2, \dots, n$, *then*

$$\|F_n - G\| \leqslant C \min\left(\frac{1}{np}, np^3\right).$$

It can be clearly seen that, in the case of identically distributed rvs with $p_k \equiv p > 0$, the accuracy of approximation is not worse than of the order $O((np)^{-1})$. The same accuracy can be expected from the symmetric case of the central limit theorem. However, our estimate holds for a stronger total variation metric. In the case of identically distributed rvs, the accuracy of approximation is at least of the order $O(n^{-1/2})$, which is achieved when $p = O(n^{-1/2})$.

Finally, nonuniform estimates are obtained for point estimates and distribution functions.

Theorem 2. *Let* $k \in \mathbb{Z}$, $k \geqslant 1$. *Then for all* $n = 2, 3 \dots,$

$$\left(1 + \frac{|k|}{\sigma}\right)\big|F_n\{k\} - G\{k\}\big| \leqslant C\varepsilon \min(\sigma^{-5}, 1),$$

$$\left(1 + \frac{|k|}{\sigma}\right)\big|F_n\big((-\infty, k]\big) - G\big((-\infty, k]\big)\big| \leqslant C\varepsilon \min(\sigma^{-4}, 1).$$

The remaining part of the paper is devoted to the proofs.

5 Auxiliary results

5.1 Inversion formulas

In this paper, the characteristic function method is used, i.e., the differences between distributions (measures) are estimated through the differences of their characteristic functions. The following lemma contains the inversion formulas used to estimate measures from their Fourier transforms.

Lemma 1. *Let $M \in M_{\mathbb{Z}}$. Then*

$$|M|_{\mathrm{K}} \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\widehat{M}(t)|}{|\mathrm{e}^{\mathrm{i}t} - 1|} \,\mathrm{d}t, \tag{5}$$

$$\|M\|_{\infty} \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \bigl|\widehat{M}(t)\bigr| \,\mathrm{d}t. \tag{6}$$

If, in addition, $\sum_{k \in \mathbb{Z}} |k||M\{k\}| < \infty$, then, for any $a \in \mathbb{R}$, $b > 0$,

$$\|M\| \leqslant (1 + b\pi)^{1/2} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \bigl|\widehat{M}(t)\bigr|^2 + \frac{1}{b^2} \bigl|\bigl(\mathrm{e}^{-\mathrm{i}ta} \widehat{M}(t)\bigr)'\bigr|^2 \,\mathrm{d}t \right)^{1/2} \tag{7}$$

and

$$|k - a|\bigl|M\{k\}\bigr| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \bigl|\bigl(\widehat{M}(t)\mathrm{e}^{-\mathrm{i}ta}\bigr)'\bigr| \,\mathrm{d}t, \tag{8}$$

$$|k - a|\bigl|M(k)\bigr| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \left(\frac{\widehat{M}(t)}{\mathrm{e}^{-\mathrm{i}t} - 1} \mathrm{e}^{-\mathrm{i}ta} \right)' \right| \,\mathrm{d}t. \tag{9}$$

Observe that (5) and (9) are trivial if integrals on the right-hand side are infinite. All inequalities are well-known and can be found in [3, Sects. 6.1, 6.2]; see, also [12] and [17, Lemma 3.3].

5.2 Heinrich's lemma

In this paper, Heinrich's lemma is used to derive the characteristic function of the sum of weakly dependent random variables.

Lemma 2. *Let $X_1, X_2, \ldots, X_k$ be 1-dependent real variables, $S_n = X_1 + X_2 + \cdots + X_n$, and let t be such that*

$$\max_{1 \leqslant k \leqslant n} \bigl(\mathbf{E}\,\bigl|\mathrm{e}^{\mathrm{i}tX_k} - 1\bigr|^2\bigr)^{1/2} \leqslant \frac{1}{6}. \tag{10}$$

Then the following representation holds: $\mathbf{E}\,\mathrm{e}^{\mathrm{i}tS_n} = \varphi_1(t)\varphi_2(t)\cdots\varphi_n(k)$. *Here* $\varphi_1(t) = \mathbf{E}\,\mathrm{e}^{\mathrm{i}tX_1}$ *and, for $k = 2, \ldots, n$,*

$$\varphi_k(t) = 1 + \mathbf{E}\,\bigl(\mathrm{e}^{\mathrm{i}tX_k} - 1\bigr) + \sum_{j=1}^{k-1} \frac{\widehat{\mathbf{E}}((\mathrm{e}^{\mathrm{i}tX_j} - 1), (\mathrm{e}^{\mathrm{i}tX_{j+1}} - 1), \ldots, (\mathrm{e}^{\mathrm{i}tX_k} - 1))}{\varphi_j(t)\varphi_{j+1}(t)\cdots\varphi_{k-1}(t)}.$$

Moreover, for $j = 1, 2, \ldots, n$,

$$\bigl|\varphi_j(t) - \mathbf{E}\,\mathrm{e}^{\mathrm{i}tX_j}\bigr| \leqslant 6 \max_{1 \leqslant k \leqslant n} \mathbf{E}\,\bigl|\mathrm{e}^{\mathrm{i}tX_k} - 1\bigr|^2.$$

Proof. The proof of Lemma 2 can be found in [6]; see also [3, pp. 208–210]. □

5.3 Explicit expression for $\mathbf{E}(Y_1, Y_2, \ldots, Y_k)$ and $\widehat{\mathbf{E}}(Y_1, Y_2, \ldots, Y_k)$

In the following two lemmas, we are going to derive the explicit expressions for $\mathbf{E}\, Y_1 \times Y_2 \cdots Y_k$ and $\widehat{\mathbf{E}} Y_1 Y_2 \cdots Y_k$ so that the expression of $\widehat{F}_n(t)$ can be later simplified.

Lemma 3. *For each $k \in \mathbb{N}$,*

$$\mathbf{E}\, Y_1 Y_2 \cdots Y_k = 2p_1 p_2 \cdots p_{k+1}(z+\bar{z})^k = 2p_1 p_2 \cdots p_{k+1}(-1)^k (z\bar{z})^k. \qquad (11)$$

Here rvs $Y_1, Y_2, \ldots$ are defined in Section 3.

Proof. The equality can be proved by induction. When $k=1$,

$$\begin{aligned}\mathbf{E}\, Y_1 &= \mathbf{E}\left(\mathrm{e}^{\mathrm{i}t\xi_1\xi_2} - 1\right) = p_1\, \mathbf{E}\left(\mathrm{e}^{\mathrm{i}t\xi_2} - 1\right) + p_1\, \mathbf{E}\left(\mathrm{e}^{-\mathrm{i}t\xi_2} - 1\right) \\ &= p_1(p_2 z + p_2\bar{z}) + p_1(p_2\bar{z} + p_2 z) = 2p_1p_2(z+\bar{z}).\end{aligned}$$

Assume that (11) equation is true for $k \in \mathbb{N}$. We must prove that the formula is true for $k+1$. Indeed,

$$\begin{aligned}&\mathbf{E}\, Y_1 Y_2 \cdots Y_{k+1} \\ &\quad= \mathbf{E}\, Y_1 \cdots Y_k\left(\mathrm{e}^{\mathrm{i}t\xi_{k+1}} - 1\right)p_{k+2} + \mathbf{E}\, Y_1 \cdots Y_k\left(\mathrm{e}^{-\mathrm{i}t\xi_{k+1}} - 1\right)p_{k+2} \\ &\quad= \mathbf{E}\, Y_1 \cdots Y_{k-1}\left(\mathrm{e}^{\mathrm{i}t\xi_k\xi_{k+1}} - 1\right)\left(\mathrm{e}^{\mathrm{i}t\xi_{k+1}} - 1\right)p_{k+2} \\ &\qquad+ \mathbf{E}\, Y_1 \cdots Y_{k-1}\left(\mathrm{e}^{\mathrm{i}t\xi_k\xi_{k+1}} - 1\right)\left(\mathrm{e}^{-\mathrm{i}t\xi_{k+1}} - 1\right)p_{k+2} \\ &\quad= \mathbf{E}\, Y_1 \cdots Y_{k-1}\left(\mathrm{e}^{\mathrm{i}t\xi_k} - 1\right)z p_{k+1}p_{k+2} \\ &\qquad+ \mathbf{E}\, Y_1 \cdots Y_{k-1}\left(\mathrm{e}^{-\mathrm{i}t\xi_k} - 1\right)\bar{z} p_{k+1}p_{k+2} \\ &\qquad+ \mathbf{E}\, Y_1 \cdots Y_{k-1}\left(\mathrm{e}^{\mathrm{i}t\xi_k} - 1\right)\bar{z} p_{k+1}p_{k+2} \\ &\qquad+ \mathbf{E}\, Y_1 \cdots Y_{k-1}\left(\mathrm{e}^{-\mathrm{i}t\xi_k} - 1\right)z p_{k+1}p_{k+2} \\ &\quad= p_{k+1}p_{k+2}(z+\bar{z})\big(\mathbf{E}\, Y_1 \cdots Y_{k-1}\left(\mathrm{e}^{\mathrm{i}t\xi_k} - 1\right) \\ &\qquad+ \mathbf{E}\, Y_1 \cdots Y_{k-1}\left(\mathrm{e}^{-\mathrm{i}t\xi_k} - 1\right)\big) \\ &\quad= p_{k+2}(z+\bar{z})\,\mathbf{E}\, Y_1 \cdots Y_k = 2p_1 \cdots p_{k+1}p_{k+2}(z+\bar{z})^{k+1}.\end{aligned}$$

It can be easily checked that

$$-z\bar{z} = -\left(\mathrm{e}^{\mathrm{i}t} - 1\right)\left(\mathrm{e}^{-\mathrm{i}t} - 1\right) = -2 + \mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t} = z + \bar{z}. \qquad \square$$

Lemma 4. *For each $k \in \mathbb{N}$, $k \geqslant 2$,*

$$\begin{aligned}&\widehat{\mathbf{E}}(Y_1, Y_2, \ldots, Y_k) \\ &\quad= 2p_1 \cdots p_{k+1}(1-2p_2)(1-2p_3)\cdots(1-2p_k)(z+\bar{z})^k \\ &\quad= 2p_1 \cdots p_{k+1}(1-2p_2)(1-2p_3)\cdots(1-2p_k)(-1)^k(z\bar{z})^k. \qquad (12)\end{aligned}$$

Here rvs Y_1, Y_2, $\ldots$ are defined in Section 3.

Proof. We apply induction. When $k = 2$, from Lemma 3 we get

$$\begin{aligned}\widehat{\mathbf{E}}(Y_1, Y_2) &= \mathbf{E}\,Y_1Y_2 - \mathbf{E}\,Y_1\,\mathbf{E}\,Y_2\\ &= 2p_1p_2p_3(z+\bar{z})^2 - 2p_1p_2(z+\bar{z})\cdot 2p_2p_3(z+\bar{z})\\ &= 2p_1p_2p_3(1-2p_2)(z+\bar{z})^2.\end{aligned}$$

Assume that (12) equation is true for $k \in \mathbb{N}$, $k \geqslant 2$. Also notice that

$$\widehat{\mathbf{E}}Y_1 = \mathbf{E}\,Y_1 = 2p_1p_2(z+\bar{z}).$$

Using Lemma 3 and induction hypothesis, it can be proved that (12) is true for $k+1$:

$$\begin{aligned}&\widehat{\mathbf{E}}(Y_1, Y_2, \dots, Y_{k+1})\\ &\quad= \mathbf{E}\,Y_1\cdots Y_{k+1} - \widehat{\mathbf{E}}Y_1\,\mathbf{E}\,Y_2\cdots Y_{k+1} - \widehat{\mathbf{E}}(Y_1,Y_2)\,\mathbf{E}\,Y_3\cdots Y_{k+1}\\ &\qquad- \sum_{j=3}^{k}\widehat{\mathbf{E}}(Y_1,\dots,Y_j)\,\mathbf{E}\,Y_{j+1}\cdots Y_{k+1}\\ &\quad= 2p_1\cdots p_{k+2}(z+\bar{z})^{k+1} - 2p_1p_2(z+\bar{z})\cdot 2p_2\cdots p_{k+2}(z+\bar{z})^k\\ &\qquad- 2p_1p_2p_3(1-2p_2)(z+\bar{z})^2\cdot 2p_3\cdots p_{k+2}(z+\bar{z})^{k-1}\\ &\qquad- \sum_{j=3}^{k}2p_1\cdots p_{j+1}(1-2p_2)\cdots(1-2p_j)(z+\bar{z})^j 2p_{j+1}\cdots p_{k+2}(z+\bar{z})^{k-j+1}\\ &\quad= 2p_1\cdots p_{k+2}(z+\bar{z})^{k+1}\\ &\qquad\times\Biggl(1-2p_2-2(1-2p_2)p_3-\sum_{j=3}^{k}2p_{j+1}(1-2p_2)\cdots(1-2p_j)\Biggr) = \cdots\\ &\quad= 2p_1\cdots p_{k+2}(z+\bar{z})^{k+1}(1-2p_2)(1-2p_3)\cdots(1-2p_{k+1}).\end{aligned}$$ □

5.4 Recursive formula for $\widehat{F}_n(t)$

Now we can easily obtain the characteristic function of the sum of 1-dependent symmetric three-point random variables.

Lemma 5. *The following representation holds for the characteristic function of S_n defined in Section 3:*

$$\widehat{F}_n(t) = \mathbf{E}\,\mathrm{e}^{\mathrm{i}tS_n} = \varphi_1(t)\varphi_2(t)\cdots\varphi_n(t).$$

Here $\varphi_1(t) = 1 + 2p_1p_2(z+\bar{z})$ and, for $k = 2, \dots, n$,

$$\begin{aligned}\varphi_k(t) &= 1 + 2p_kp_{k+1}(z+\bar{z})\\ &\quad+ \sum_{j=1}^{k-1}\frac{2p_jp_{j+1}\cdots p_{k+1}(1-2p_{j+1})\cdots(1-2p_k)(z+\bar{z})^{k-j+1}}{\varphi_j(t)\varphi_{j+1}(t)\cdots\varphi_{k-1}(t)}.\end{aligned}$$

Proof. First, note that if $p \leqslant 1/24$, then (10) is satisfied for all t. Indeed,

$$\begin{aligned}\mathbf{E}\left|\mathrm{e}^{\mathrm{i}tX_k} - 1\right|^2 &= \mathbf{E}\left|\mathrm{e}^{\mathrm{i}t\xi_k\xi_{k+1}} - 1\right|^2 \\ &= \left|\mathrm{e}^{\mathrm{i}t} - 1\right|^2 2p_k p_{k+1} + \left|\mathrm{e}^{-\mathrm{i}t} - 1\right|^2 2p_k p_{k+1} \\ &\leqslant |z|^2 4p^2 \leqslant 2^2 \cdot 4 \cdot \frac{1}{24^2} \leqslant \frac{1}{36}.\end{aligned}$$

Hence,

$$\max_{1\leqslant k\leqslant n}\left(\mathbf{E}\left|\mathrm{e}^{\mathrm{i}tX_k} - 1\right|^2\right)^{1/2} \leqslant \frac{1}{6}.$$

It remains to apply Lemmas 2, 3, and 4. □

5.5 Estimates for $\varphi_k(t)$

Now $\varphi_k(t)$ can be estimated.

Lemma 6. *For all t, $k = 2, \dots, n$, $n \geqslant 3$, the following estimates hold*:

$$\left|\varphi_k(t) - 1\right| \leqslant 3p^2|z|^2, \qquad \frac{1}{|\varphi_k(t)|} \leqslant \frac{48}{47}, \tag{13}$$

$$\left|\varphi_k(t) - 1 - 2p_k p_{k+1}(z + \bar{z})\right| \leqslant 3|z|^4 p_{k-1}p_k p_{k+1}, \tag{14}$$

$$\left|\varphi_k(t)\right| \leqslant 1 - 6p_k p_{k+1}\sin^2\frac{t}{2}. \tag{15}$$

Here functions $\varphi_k(t)$ are defined in Lemma 5.

Proof. We prove (13) by induction. Notice that $|z + \bar{z}| = |-z\bar{z}| = |z|^2 \leqslant 4$. When $k = 2$,

$$\begin{aligned}\left|\varphi_2(t) - 1\right| &= 2p_2p_3\left|(z + \bar{z})\right| + \left|\frac{2p_1p_2p_3(1 - 2p_2)(z + \bar{z})^2}{1 + 2p_1p_2(z + \bar{z})}\right| \\ &\leqslant 2p^2|z|^2 + \frac{2p^3|z|^4}{1 - 8p^2} \leqslant 2p^2|z|^2 + \frac{24}{71}p^2|z|^2 \leqslant 3p^2|z|^2.\end{aligned}$$

Assume that $|\varphi_j(t) - 1| \leqslant 3p^2|z|^2$ is correct for $j = 3, \dots, k-1$. Taking into account the trivial estimate $|z| \leqslant 2$, we obtain that, for all $j = 3, \dots, k-1$,

$$\begin{aligned}\left|\varphi_j(t) - 1\right| &\leqslant 3p^2|z|^2 \leqslant \frac{12}{24^2}, \\ \left|\varphi_j(t)\right| = \left|1 - \left(1 - \varphi_j(t)\right)\right| &\geqslant 1 - \left|1 - \varphi_j(t)\right| \geqslant 1 - \frac{12}{24^2} = \frac{47}{48}, \\ \frac{1}{|\varphi_j(t)|} &\leqslant \frac{48}{47},\end{aligned}$$

and, using the expression of $\varphi_k(t)$ from Lemma 5, we get

$$\begin{aligned}
\left|\varphi_k(t)-1\right| &\leqslant 2p_kp_{k+1}|z|^2 \\
&\quad+\sum_{j=1}^{k-1}\frac{2p_jp_{j+1}\dots p_{k+1}(1-2p_{j+1})\cdots(1-2p_k)|z|^{2(k-j+1)}}{|\varphi_j(t)||\varphi_{j+1}(t)|\cdots|\varphi_{k-1}(t)|} \\
&\leqslant 2p^2|z|^2+2p^2|z|^2\sum_{j=1}^{k-1}\left(\frac{48p|z|^2}{47}\right)^{k-j} \\
&\leqslant 2p^2|z|^2\left\{1+\frac{48\cdot 4}{47\cdot 24}+\left(\frac{48\cdot 4}{47\cdot 24}\right)^2+\cdots\right\} \\
&\leqslant 3p^2|z|^2.
\end{aligned}$$

Inequality (14) is proved analogously. When $k=2$,

$$\begin{aligned}
&\left|\varphi_2(t)-1-2p_2p_3(z+\bar z)\right| \\
&\quad=\left|\frac{2p_1p_2p_3(1-2p_2)(z+\bar z)^2}{1+2p_1p_2(z+\bar z)}\right|\leqslant\frac{2}{1-8p^2}|z|^4p_1p_2p_3 \\
&\quad\leqslant 2|z|^4p_1p_2p_3.
\end{aligned}$$

Assume that $|\varphi_j(t)-1-2p_jp_{j+1}(z+\bar z)|\leqslant 2|z|^4p_{j-1}p_jp_{j+1}$ holds for $j=3,\dots,k-1$. Then

$$\begin{aligned}
&\left|\varphi_k(t)-1-2p_kp_{k+1}(z+\bar z)\right| \\
&\quad\leqslant\sum_{j=1}^{k-1}\frac{2p_jp_{j+1}\cdots p_{k+1}(1-2p_{j+1})\cdots(1-2p_k)|z|^{2(k-j+1)}}{|\varphi_j(t)||\varphi_{j+1}(t)|\cdots|\varphi_{k-1}(t)|} \\
&\quad\leqslant\sum_{j=1}^{k-1}2p_jp_{j+1}\cdots p_{k+1}|z|^{2(k-j+1)}\left(\frac{48}{47}\right)^{k-1-j+1} \\
&\quad\leqslant 2|z|^4p_{k-1}p_kp_{k+1}\frac{48}{47}\sum_{j=1}^{k-1}\left(\frac{48p|z|^2}{47}\right)^{k-j-1} \\
&\quad\leqslant 2|z|^4p_{k-1}p_kp_{k+1}\frac{48}{47}\sum_{j=1}^{k-1}\left(\frac{8}{47}\right)^{k-j-1} \\
&\quad\leqslant 3|z|^4p_{k-1}p_kp_{k+1}.
\end{aligned}$$

The proof of (15) requires the application of estimate (14):

$$\begin{aligned}
\left|\varphi_k(t)\right| &= \left|1+2p_kp_{k+1}(z+\bar z)+\varphi_k(t)-1-2p_kp_{k+1}(z+\bar z)\right| \\
&\leqslant\left|1+2p_kp_{k+1}(z+\bar z)\right|+3|z|^4p_{k-1}p_kp_{k+1} \\
&\leqslant\left|1+2p_kp_{k+1}(z+\bar z)\right|+0.5|z|^2p_kp_{k+1}.
\end{aligned}$$

Since $z+\bar z=-4\sin^2(t/2)$ and $|z|^2=4\sin^2(t/2)$, we get

$$\begin{aligned}\left|\varphi_k(t)\right| &\leqslant \left|1-8p_kp_{k+1}\sin^2\frac{t}{2}\right|+2p_kp_{k+1}\sin^2\frac{t}{2}\\ &\leqslant 1-6p_kp_{k+1}\sin^2\frac{t}{2}.\end{aligned}$$ □

Since ξ_j $(j=1,2,\dots)$ are independent, observe that

$$\mathbf{E}\,S_n=\sum_{j=1}^n\mathbf{E}\,\xi_j\xi_{j+1}=\sum_{j=1}^n\mathbf{E}\,\xi_j\,\mathbf{E}\,\xi_{j+1}=0,$$

and

$$\sigma^2=\mathbf{Var}\,S_n=\sum_{j=1}^n\mathbf{E}(\xi_j\xi_{j+1})^2=\sum_{j=1}^n\mathbf{E}\,\xi_j^2\,\mathbf{E}\,\xi_{j+1}^2=4\sum_{j=1}^n p_jp_{j+1}.$$

Hence,

$$\left|\widehat{F}_n(t)\right|\leqslant\prod_{k=1}^n\exp\left\{-6p_kp_{k+1}\sin^2\frac{t}{2}\right\}\leqslant\exp\left\{-\frac{3}{2}\sigma^2\sin^2\frac{t}{2}\right\}.$$

5.6 Estimates for $\varphi_k'(t)$

To approximate lattice variables in the total variation metric, estimates for the derivatives of characteristic functions are needed.

Lemma 7. *For all t, $k=2,\dots,n$, $n\geqslant 3$, the following estimates hold*:

$$\left|\varphi_k'(t)\right|\leqslant 6p_kp_{k+1}|z|,\qquad \left|\varphi_k'(t)-2p_kp_{k+1}(z+\bar z)'\right|\leqslant 12p_{k-1}p_kp_{k+1}|z|^3,$$

$$\begin{aligned}&\left|\varphi_k'(t)-2p_kp_{k+1}(z+\bar z)'-2p_{k-1}p_kp_{k+1}(1-2p_k)\bigl((z+\bar z)^2\bigr)'\right|\\ &\quad\leqslant 2p_{k-1}p_kp_{k+1}|z|^4.\end{aligned}$$

Here functions $\varphi_k(t)$ are defined in Lemma 5.

Proof. Observe that

$$|z'|=1,\qquad|\bar z'|=1,\qquad|(z+\bar z)'|\leqslant 2|z|,\qquad\left|\bigl((z+\bar z)^2\bigr)'\right|\leqslant 4|z|^3.$$

The first two estimates follow from the last one. Indeed,

$$\begin{aligned}&\left|\varphi_k'(t)-2p_kp_{k+1}(z+\bar z)'\right|\\ &\quad\leqslant 2p_{k-1}p_kp_{k+1}|z|^4+\left|2p_{k-1}p_kp_{k+1}(1-2p_k)\bigl((z+\bar z)^2\bigr)'\right|\\ &\quad\leqslant 2p_{k-1}p_kp_{k+1}|z|^4+2p_{k-1}p_kp_{k+1}\cdot 4|z|^3\\ &\quad\leqslant 12p_{k-1}p_kp_{k+1}|z|^3;\\ &\left|\varphi_k'(t)\right|\leqslant 12p_{k-1}p_kp_{k+1}|z|^3+|2p_kp_{k+1}(z+\bar z)'|\\ &\qquad\qquad\leqslant 6p_kp_{k+1}|z|.\end{aligned}$$

Therefore, it suffices to prove the last estimate. We use induction. It is easy to check that it holds for $k = 2$:

$$\begin{aligned}
&\bigl|\varphi_2'(t) - 2p_2p_3(z+\bar z)' - 2p_1p_2p_3(1-2p_2)\bigl((z+\bar z)^2\bigr)'\bigr| \\
&\quad= \left|\left(\frac{2p_1p_2p_3(1-2p_2)(z+\bar z)^2}{\varphi_1(t)}\right)' - 2p_1p_2p_3(1-2p_2)\bigl((z+\bar z)^2\bigr)'\right| \\
&\quad\leqslant 2p_1p_2p_3\left|\frac{((z+\bar z)^2)'}{\varphi_1(t)} - \frac{\varphi_1'(t)((z+\bar z))^2}{\varphi_1^2(t)} - \bigl((z+\bar z)^2\bigr)'\right| \\
&\quad\leqslant 2p_1p_2p_3\left(\frac{|((z+\bar z)^2)'||\varphi_1(t)-1|}{|\varphi_1(t)|} + \frac{|(z+\bar z)^2||\varphi_1'(t)|}{|\varphi_1(t)|^2}\right) \\
&\quad\leqslant 2p_1p_2p_3\left(\frac{4|z|^3|2p_1p_2(z+\bar z)|}{|1+2p_1p_2(z+\bar z)|} + \frac{|(z+\bar z)^2||2p_1p_2(z+\bar z)'|}{|1+2p_1p_2(z+\bar z)|^2}\right) \\
&\quad\leqslant 0.09p_1p_2p_3|z|^4.
\end{aligned}$$

Next, assume that it also holds for $j = 3, \dots, k-1$. Then

$$\begin{aligned}
&\left|\varphi_k'(t) - 2p_kp_{k+1}(z+\bar z)' - \left(\frac{2p_{k-1}p_kp_{k+1}(1-2p_k)(z+\bar z)^2}{\varphi_{k-1}(t)}\right)'\right| \\
&\quad\leqslant \left|\sum_{j=1}^{k-2}\frac{2(k-j+1)p_j\cdots p_{k+1}(1-2p_{j+1})\cdots(1-2p_k)(z+\bar z)^{k-j}(z+\bar z)'}{\varphi_j(t)\cdots\varphi_{k-1}(t)}\right. \\
&\qquad\left.-\sum_{j=1}^{k-2}\frac{2p_j\cdots p_{k+1}(1-2p_{j+1})\cdots(1-2p_k)(z+\bar z)^{k-j+1}}{\varphi_j(t)\cdots\varphi_{k-1}(t)}\sum_{m=j}^{k-1}\frac{\varphi_m'(t)}{\varphi_m(t)}\right| \\
&\quad\leqslant 2\cdot 4p_{k-1}p_kp_{k+1}\frac{1}{24}|z+\bar z|^2\left(\frac{48}{47}\right)^2\sum_{j=1}^{k-2}(k-j+1)\left(4p\frac{48}{47}\right)^{k-j-2} \\
&\qquad+ 2p_{k-1}p_kp_{k+1}\frac{1}{24}|z+\bar z|^3\left(\frac{48}{47}\right)^3 6p_kp_{k+1}|z|\sum_{j=1}^{k-2}(k-j)\left(4p\frac{48}{47}\right)^{k-j-2} \\
&\quad\leqslant 4p_{k-1}p_kp_{k+1}\frac{1}{24}|z|^4\left(\frac{48}{47}\right)^2\left[2\sum_{j=0}^{\infty}(j+3)\left(\frac{8}{47}\right)^j + \frac{2}{47}\sum_{j=0}^{\infty}(j+2)\left(\frac{8}{47}\right)^j\right] \\
&\quad\leqslant 2p_{k-1}p_kp_{k+1}|z|^4.
\end{aligned}$$

The last inequality has been obtained using the fact that, for $0 < x < 1$,

$$\sum_{j=0}^{\infty}x^j = \frac{1}{1-x}, \qquad \sum_{j=0}^{\infty}jx^j = \frac{x}{(1-x)^2}.$$

To complete the proof, we observe that

$$
\begin{aligned}
&2p_{k-1}p_kp_{k+1}(1-2p_k)\left|\left(\frac{(z+\bar z)^2}{\varphi_{k-1}(t)}\right)' - \left((z+\bar z)^2\right)'\right| \\
&\leqslant 2p_{k-1}p_kp_{k+1}\frac{|((z+\bar z)^2)'||\varphi_{k-1}(t)-1|}{|\varphi_{k-1}(t)|} 2p_{k-1}p_kp_{k+1}\frac{|(z+\bar z)^2||\varphi_{k-1}'(t)|}{|\varphi_{k-1}(t)|^2} \\
&\leqslant 0.2p_{k-1}p_kp_{k+1}|z|^4.
\end{aligned}
$$

□

5.7 Closeness of $\widehat{F}_n(t)$ and $\widehat{G}(t)$

Now we can approximate $\widehat{F}_n(t)$. The well-known inequalities are used:

$$
x\mathrm{e}^{-x} \leqslant 1, \quad x>0; \qquad 1+x\leqslant \mathrm{e}^x, \quad x\in\mathbb{R};
$$

$$
\left|\prod_{j=1}^n \varphi_j(t) - \prod_{j=1}^n \widehat{G}_j(t)\right| \leqslant \sum_{j=1}^n \left|\varphi_j(t)-\widehat{G}_j(t)\right| \prod_{k=1}^{j-1}\left|\varphi_k(t)\right| \prod_{k=j+1}^{n}\left|\widehat{G}_k(t)\right|.
$$

Observe that $\widehat{G}(t)=\prod_{k=1}^n \widehat{G}_k(t)$ and

$$
\widehat{G}_k(t) = \exp\left\{2p_kp_{k+1}\left(\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t}-2\right)\right\} = \exp\left\{-8p_kp_{k+1}\sin^2\frac{t}{2}\right\}.
$$

Lemma 8. *For all* $n\geqslant 3$, $|t|\leqslant\pi$, *we have*

$$
\begin{aligned}
\left|\widehat{F}_n(t)-\widehat{G}(t)\right| &\leqslant C\varepsilon\sin^4\frac{t}{2}\exp\left\{-C\sigma^2\sin^2\frac{t}{2}\right\} \leqslant C\varepsilon\min\left(\frac{\varepsilon}{\sigma^4},1\right), \\
\left|(\widehat{F}_n(t)-\widehat{G}(t))'\right| &\leqslant C\varepsilon\sin^3\frac{t}{2}\exp\left\{-C\sigma^2\sin^2\frac{t}{2}\right\} \leqslant C\varepsilon\min\left(\frac{\varepsilon}{\sigma^3},1\right).
\end{aligned}
$$

Proof. We know that, for any finite signed measure M and $s\in\mathbb{Z}_+$,

$$
\exp\{M\} = I + \sum_{j=1}^s \frac{M^j}{j!} + \frac{1}{(s+1)!}\|M\|^{s+1}\exp\left\{\|M\|\right\}\varTheta, \quad \|\varTheta\|\leqslant 1. \tag{16}
$$

The analogue of (16) for Fourier transforms gives us

$$
\begin{aligned}
\widehat{G}_k(t) &= 1 + 2p_kp_{k+1}(z+\bar z) + C\theta(p_kp_{k+1})^2\sin^4\frac{t}{2}, \\
\widehat{G}_k'(t) &= 2p_kp_{k+1}(z+\bar z)' + C\theta(p_kp_{k+1})^2\sin^3\frac{t}{2}.
\end{aligned}
$$

From Lemma 6 it follows that

$$
\begin{aligned}
\left|\varphi_k(t)-\widehat{G}_k(t)\right| &\leqslant \left|\varphi_k(t)-1-2p_kp_{k+1}(z+\bar z)\right| \\
&\quad + \left|\widehat{G}_k(t)-1-2p_kp_{k+1}(z+\bar z)\right| \\
&\leqslant 3|z|^4 p_{k-1}p_kp_{k+1} + C(p_kp_{k+1})^2\sin^4\frac{t}{2},
\end{aligned}
$$

and hence

$$
\begin{aligned}
\left|\widehat{F}_n(t)-\widehat{G}(t)\right| &= \left|\prod_{j=1}^{n}\varphi_j(t)-\prod_{j=1}^{n}\widehat{G}_j(t)\right| \\
&\leqslant \sum_{j=1}^{n}\left|\varphi_j(t)-\widehat{G}_j(t)\right|\prod_{k=1}^{j-1}\left(1-6p_kp_{k+1}\sin^2\frac{t}{2}\right) \\
&\quad\times\prod_{k=j+1}^{n}\exp\left\{-8p_kp_{k+1}\sin^2\frac{t}{2}\right\} \\
&\leqslant \sum_{j=1}^{n}\frac{|\varphi_j(t)-\widehat{G}_j(t)|}{1-6p_jp_{j+1}\sin^2\frac{t}{2}}\prod_{k=1}^{n}\exp\left\{-6p_kp_{k+1}\sin^2\frac{t}{2}\right\} \\
&\leqslant C\varepsilon\sin^4\frac{t}{2}\exp\left\{-\frac{3}{2}\sigma^2\sin^2\frac{t}{2}\right\} \\
&\leqslant C\varepsilon\min\left(\frac{1}{\sigma^4},1\right).
\end{aligned}
$$

Similarly, from Lemma 7 it follows that

$$
\begin{aligned}
\left|\varphi_k'(t)-\widehat{G}_k'(t)\right| &\leqslant \left|\varphi_k'(t)-2p_kp_{k+1}(z+\bar{z})'\right| \\
&\quad+\left|\widehat{G}_k'(t)-2p_kp_{k+1}(z+\bar{z})'\right| \\
&\leqslant 12p_{k-1}p_kp_{k+1}|z|^3+C(p_kp_{k-1})^2\sin^3\frac{t}{2},
\end{aligned}
$$

and hence

$$
\begin{aligned}
\left|(\widehat{F}_n(t)-\widehat{G}(t))'\right| &\leqslant \sum_{k=1}^{n}\left|\varphi_k'(t)\right|\left|\prod_{m\neq k}^{n}\varphi_m(t)-\prod_{m\neq k}^{n}\widehat{G}_m(t)\right| \\
&\quad+\sum_{k=1}^{n}\left|\varphi_k'(t)-\widehat{G}_k'(t)\right|\prod_{m\neq k}^{n}\left|\widehat{G}_m(t)\right| \\
&\leqslant C\sum_{k=1}^{n}6p_kp_{k+1}|z|\varepsilon\sin^4\frac{t}{2}\exp\left\{\sum_{k=1}^{n}-6p_kp_{k+1}\sin^2\frac{t}{2}\right\} \\
&\quad+C\sum_{k=1}^{n}\left(p_{k-1}p_kp_{k+1}+(p_{k-1}p_k)^2\right)\sin^3\frac{t}{2} \\
&\quad\times\exp\left\{-\sum_{k=1}^{n}8p_kp_{k+1}\sin^2\frac{t}{2}\right\} \\
&\leqslant C\varepsilon\sin^3\frac{t}{2}\exp\left\{-C\sigma^2\sin^2\frac{t}{2}\right\} \\
&\leqslant C\varepsilon\min\left(\frac{1}{\sigma^3},1\right).
\end{aligned}
$$

□

6 Proofs of the main results

Proof of Theorem 1. For the proof the following inequality is used:

$$\int_{-\pi}^{\pi} \left|\sin\frac{t}{2}\right|^k \exp\left\{-2\lambda\sin^2\frac{t}{2}\right\}\mathrm{d}t \leqslant C(k)\lambda^{-(k+1)/2}, \quad k=0,1,2,\dots,\ \lambda>0.$$

Using the inversion formula (6) and Lemma 8, we get

$$\begin{aligned}\|F_n-G\|_\infty &\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\widehat{F}_n(t)-\widehat{G}(t)\right|\mathrm{d}t\\ &\leqslant \int_{-\pi}^{\pi} C\varepsilon\sin^4\frac{t}{2}\exp\left\{-C\sigma^2\sin^2\frac{t}{2}\right\}\mathrm{d}t\\ &\leqslant C\varepsilon\min\left(\frac{1}{\sigma^5},1\right).\end{aligned}$$

Applying the inversion formula (7) with $a=0$, $b=1$ and Lemma 8, we get

$$\begin{aligned}\|F_n-G\| &\leqslant (1+\pi)^{1/2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\widehat{F}_n(t)-\widehat{G}(t)\right|^2+\left|\left(\widehat{F}_n(t)-\widehat{G}(t)\right)'\right|^2\mathrm{d}t\right)^{1/2}\\ &\leqslant (1+\pi)^{1/2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\left(C\varepsilon\sin^4\frac{t}{2}\exp\left\{-C\sigma^2\sin^2\frac{t}{2}\right\}\right)^2\right.\right.\\ &\quad\left.\left.+\left(C\varepsilon\sin^3\frac{t}{2}\exp\left\{-C\sigma^2\sin^2\frac{t}{2}\right\}\right)^2\right]\mathrm{d}t\right)^{1/2}\\ &\leqslant C\varepsilon\min\left(\frac{1}{\sigma^4},1\right).\end{aligned}$$

□

Proof of Theorem 2. Applying the inversion formula (8) with $a=0$ and Lemma 8, we get

$$\begin{aligned}|k|\left|F_n\{k\}-G\{k\}\right| &\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left(\widehat{F}_n(t)-\widehat{G}(t)\right)'\right|\mathrm{d}t\\ &\leqslant C\int_{-\pi}^{\pi}\varepsilon\sin^3\frac{t}{2}\exp\left\{-C\sigma^2\sin^2\frac{t}{2}\right\}\mathrm{d}t\\ &\leqslant C\varepsilon\min\left(\frac{1}{\sigma^4},1\right).\end{aligned}$$

Combining the obtained nonuniform estimate with the local norm estimate we get

$$\bigl|F_n\{k\} - G\{k\}\bigr| \leqslant \|F_n - G\|_\infty \leqslant C\varepsilon \min\biggl(\frac{1}{\sigma^5}, 1\biggr).$$

Therefore, we can write the following nonuniform estimate:

$$\biggl(1 + \frac{|k|}{\sigma}\biggr)\bigl|F_n\{k\} - G\{k\}\bigr| \leqslant C\varepsilon \min\biggl(\frac{1}{\sigma^5}, 1\biggr).$$

The inversion formula (9) with $a = 0$ leads to

$$\begin{aligned}
&|k|\bigl|F_n\bigl((-\infty, k]\bigr) - G\bigl((-\infty, k]\bigr)\bigr| \\
&\quad\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \biggl|\biggl(\frac{\widehat{F}_n(t) - \widehat{G}(t)}{\mathrm{e}^{-\mathrm{i}t} - 1}\biggr)'\biggr| \,\mathrm{d}t \\
&\quad\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \biggl|\frac{(\widehat{F}_n(t) - \widehat{G}(t))'}{\mathrm{e}^{-\mathrm{i}t} - 1}\biggr| \,\mathrm{d}t + \frac{1}{2\pi}\int_{-\pi}^{\pi} \biggl|\frac{(\widehat{F}_n(t) - \widehat{G}(t))(\mathrm{e}^{-\mathrm{i}t} - 1)'}{(\mathrm{e}^{-\mathrm{i}t} - 1)^2}\biggr| \,\mathrm{d}t \\
&\quad\leqslant C\int_{-\pi}^{\pi} \frac{\varepsilon \sin^3\frac{t}{2} \exp\{-C\sigma^2 \sin^2\frac{t}{2}\}}{2|\sin\frac{t}{2}|} \,\mathrm{d}t + C\int_{-\pi}^{\pi} \frac{\varepsilon \sin^4\frac{t}{2} \exp\{-C\sigma^2 \sin^2\frac{t}{2}\}}{4\sin^2\frac{t}{2}} \,\mathrm{d}t \\
&\quad\leqslant C\varepsilon \min\biggl(\frac{1}{\sigma^3}, 1\biggr).
\end{aligned}$$

Since

$$\begin{aligned}
\bigl|F_n\bigl((-\infty, k]\bigr) - G\bigl((-\infty, k]\bigr)\bigr| &\leqslant |F_n - G|_{\mathrm{K}} \leqslant \|F_n - G\| \\
&\leqslant C\varepsilon \min\biggl(\frac{1}{\sigma^4}, 1\biggr),
\end{aligned}$$

the nonuniform estimate can be expressed as

$$\biggl(1 + \frac{|k|}{\sigma}\biggr)\bigl|F_n\bigl((-\infty, k]\bigr) - G\bigl((-\infty, k]\bigr)\bigr| \leqslant C\varepsilon \min\biggl(\frac{1}{\sigma^4}, 1\biggr). \qquad \square$$

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