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Joint universality of periodic zeta-functions with multiplicative coefficients

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Abstract. The periodic zeta-function is defined by the ordinary Dirichlet series with periodic coefficients. In the paper, joint universality theorems on the approximation of a collection of analytic functions by nonlinear shifts of periodic zeta-functions with multiplicative coefficients are obtained. These theorems do not use any independence hypotheses on the coefficients of zeta-functions.

Keywords: joint universality, periodic zeta-function, space of analytic functions, weak convergence.

1 Introduction

After a famous Voronin's work [27], it is known that the majority of classical zeta- and L-functions have the universality property, i.e., they approximate wide classes of analytic functions. Voronin obtained the universality property for the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad s = \sigma + \mathrm{i}t,\ \sigma > 1,$$

which has meromorphic continuation to the whole complex plane with unique simple pole at the point $s = 1$ with residue 1. Let $D = \{s \in \mathbb{C}\colon 1/2 < \sigma < 1\}$. Voronin considered approximation of analytic functions defined on D by shifts $\zeta(s + \mathrm{i}\tau)$, $\tau \in \mathbb{R}$. For the last version of the Voronin universality theorem, it is convenient to use the following notation. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements,

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and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous nonvanishing functions on K that are analytic in the interior of K. Moreover, let $\operatorname{meas} A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the Voronin theorem asserts that if $K \in \mathcal{K}$ and $f(s) \in H_0(K)$, then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Bigl\{\tau \in [0,T]\colon \sup_{s\in K} \bigl|\zeta(s+\mathrm{i}\tau) - f(s)\bigr| < \varepsilon\Bigr\} > 0.$$

A proof of the above statement by different methods is given in [1, 6], see also [13, 25].

A similar assertion is obtained for Dirichlet L-functions [1, 6, 11, 27]

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \sigma > 1,$$

where χ is a Dirichlet character.

More general there are zeta-functions attached to certain cusp forms F

$$\zeta(s,F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}, \quad \sigma > \frac{\kappa+1}{2},$$

where $c(m)$ are Fourier coefficients of the form F, and κ denotes the weight of F. Also, the functions $\zeta(s,F)$ has analytic continuation to an entire function. The universality for $\zeta(s,F)$ with normalized Hecke eigen cusp forms was obtained in [19].

The above mentioned zeta-functions have a one common feature, they have the Euler product over prime numbers. For example,

$$\zeta(s,F) = \prod_p \biggl(1 - \frac{\alpha(p)}{p^s}\biggr)^{-1} \biggl(1 - \frac{\beta(p)}{p^s}\biggr)^{-1},$$

where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers such that $c(p) = \alpha(p) + \beta(p)$, and p denotes a prime number.

A nonclassical generalization of the functions $\zeta(s)$ and $L(s,\chi)$ is the so-called periodic zeta-function with multiplicative coefficients. Let $\mathfrak{a} = \{a_m\colon m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. Obviously, there exists a constant $c = c(\mathfrak{a}) > 0$ such that $|a_m| \leqslant c$ for all $m \in \mathbb{N}$. The periodic zeta-function $\zeta(s;\mathfrak{a})$ is defined by the Dirichlet series

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

which is absolutely convergent for $\sigma > 1$.

In virtue of the periodicity of $\mathfrak{a}$, the equality

$$\zeta(s;\mathfrak{a}) = \frac{1}{q^s} \sum_{l=1}^{q} a_l \zeta\biggl(s, \frac{l}{q}\biggr) \tag{1}$$

holds, where $\zeta(s,\alpha)$ is the classical Hurwitz zeta-function with parameter $0<\alpha\leqslant 1$ that has, as $\zeta(s)$, meromorphic continuation to the whole complex plane with unique simple pole at the point $s=1$ with residue 1. Thus, the function $\zeta(s;\mathfrak{a})$ can be analytically continued to the whole complex plane, except for a simple pole at the point $s=1$ with residue

$$r_{\mathfrak{a}} \stackrel{\text{def}}{=} \frac{1}{q}\sum_{l=1}^{q} a_l.$$

If $r_{\mathfrak{a}}=0$, then $\zeta(s;\mathfrak{a})$ is an entire function.

Bagchi obtained [1] the universality of the function

$$\zeta_1(s;\mathfrak{a}) = \sum_{\substack{m=1\\(m,q)=1}}^{\infty} \frac{a_m}{m^s}, \quad \sigma>1.$$

Steuding [24, 25] considered the function $\zeta(s;\mathfrak{a})$ with nonmultiplicative sequence $\mathfrak{a}$ and proved its universality. The paper [20] is devoted to the universality of $\zeta(s;\mathfrak{a})$ with multiplicative $\mathfrak{a}$ ($a_{mn}=a_m a_n$ for coprimes m and n, and $a_1=1$). If the sequence $\mathfrak{a}$ is multiplicative, then the function $\zeta(s;\mathfrak{a})$ has the Euler product, i.e., for $\sigma>1$,

$$\zeta(s;\mathfrak{a}) = \prod_p \left(1+\sum_{k=1}^{\infty}\frac{a_{p^k}}{p^{ks}}\right).$$

Kaczorowski [10] introduced new restricted type of universality for $\zeta(s;\mathfrak{a})$ involving the notion of height of the set K.

Zeta- and L-functions also have a joint universality property. In this case, a collection of analytic functions is approximated simultaneously by a collection of shifts of zeta- or L-functions. The first joint universality results were obtained for Dirichlet L-functions in [1, 2, 6, 26], see also [11, 15, 25]. It is clear that, in the case of joint universality, the approximating shifts must be in some sense independent. In the case of Dirichlet L-functions, the nonequivalence of Dirichlet characters is used (two Dirichlet characters are called equivalent if they are generated by the same primitive characters). The joint universality Voronin theorem [26] says that if $\chi_1,\dots,\chi_r$ are pairwise nonequivalent Dirichlet characters, for $j=1,\dots,r$, $K_j\in\mathcal{K}$ and $f_j(s)\in H_0(K_j)$, then, for every $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\Bigl\{\tau\in[0,T]\colon \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\bigl|L(s+\mathrm{i}\tau;\chi_j)-f_j(s)\bigr|<\varepsilon\Bigr\}>0.$$

Pańkowski in [23] proposed a new way of joint universality for Dirichlet L-functions by using different shifts for L-functions with arbitrary characters $\chi_1,\dots,\chi_r$. Let $\alpha_1,\dots,\alpha_r\in\mathbb{R}$, $a_1,\dots,a_r\in\mathbb{R}^+$, and $b_1,\dots,b_r$ be such that

$$b_j\in\begin{cases}\mathbb{R} & \text{if } a_j\notin\mathbb{N},\\ (-\infty,0]\cup(1+\infty) & \text{if } a_j\in\mathbb{N},\end{cases}$$

and $a_j \neq a_k$ or $b_j \neq b_k$ if $k \neq j$. Moreover, let $K \in \mathcal{K}$, $f_1, \dots, f_r \in H_0(K)$. Then the Pańkowski theorem asserts that, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Bigl\{\tau \in [2,T]\colon \sup_{1\leqslant j\leqslant r} \sup_{s\in K} \bigl|L\bigl(s + \mathrm{i}\alpha_j \tau^{a_j} \log^{b_j}\tau; \chi_j\bigr) - f_j(s)\bigr| < \varepsilon\Bigr\} > 0.$$

Other joint universality results can be found in the excellent survey paper [21].

The present paper is devoted to the joint universality for periodic zeta-functions. Suppose that, for $j = 1, \dots, r$, $\mathfrak{a}_j = \{a_{jm}\colon m \in \mathbb{N}\}$ is a periodic sequence of complex numbers with minimal period $q_j \in \mathbb{N}$. Denote by q the least common multiple of the periods $q_1, \dots, q_r$, by $l_1, \dots, l_{r_1}$ ($r_1 = \varphi(q)$ is the Euler totient function) the reduced system modulo q, and define the matrix

$$A = \begin{pmatrix} a_{1l_1} & a_{2l_1} & \dots & a_{rl_1} \\ a_{1l_2} & a_{2l_2} & \dots & a_{rl_2} \\ \dots & \dots & \dots & \dots \\ a_{1l_{r_1}} & a_{2l_{r_1}} & \dots & a_{rl_{r_1}} \end{pmatrix}.$$

Then, in [18], the following joint universality theorem has been proved.

Theorem 1. *Suppose that the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are multiplicative and* $\operatorname{rank} A = r$. *For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Bigl\{\tau \in [0,T]\colon \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \bigl|\zeta(s + \mathrm{i}\tau; \mathfrak{a}_j) - f_j(s)\bigr| < \varepsilon\Bigr\} > 0.$$

To be precise, in [18], a technical condition

$$\sum_{k=1}^{\infty} \frac{|a_{jp^k}|}{p^{k/2}} \leqslant c_j < 1, \quad j = 1, \dots, r,$$

was required, however, it can be easily removed.

Joint universality of more general collections of zeta-functions was studied in [12, 14, 16, 17] and [7–9]. We note that joint mixed universality theorems imply those for zeta-function with Euler product.

The aim of this paper is to replace the condition $\operatorname{rank} A = r$ in Theorem 1 by using more general, nonlinear shifts $\zeta(s + \mathrm{i}\gamma_j(\tau); \mathfrak{a}_j)$, with some functions $\gamma_j(\tau)$. In [18], the linear shifts $\zeta(s + \mathrm{i}\tau; \mathfrak{a}_j)$ were used. We propose two types of $\gamma_j(\tau)$.

Denote by $U_1(T_0)$, $T_0 > 0$, the class of real increasing to ∞ continuously differentiable functions $\gamma(\tau)$ with monotonic derivative $\gamma'(\tau)$ on $[T_0, \infty)$ such that $\gamma(2\tau) \times \max_{\tau\leqslant u\leqslant 2\tau} 1/\gamma'(u) \ll \tau$ as $\tau \to \infty$.

Theorem 2. *Suppose that the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are multiplicative, $a_1, \dots, a_r$ are real algebraic numbers linearly independent over the field of rational numbers $\mathbb{Q}$, and $\gamma(\tau) \in U_1(T_0)$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T - T_0} \operatorname{meas}\Bigl\{\tau \in [T_0,T]\colon \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \bigl|\zeta\bigl(s + \mathrm{i}a_j\gamma(\tau); \mathfrak{a}_j\bigr) - f_j(s)\bigr| < \varepsilon\Bigr\} > 0.$$

Moreover, the limit

$$\lim_{T\to\infty}\frac{1}{T-T_0}\,\mathrm{meas}\Big\{\tau\in[T_0,T]\colon \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\big|\zeta\big(s+\mathrm{i}a_j\gamma(\tau);\mathfrak{a}_j\big)-f_j(s)\big|<\varepsilon\Big\}>0$$

exists for all but at most countably many $\varepsilon>0$.

Denote by $U_r(T_0)$ the class of real increasing to infinity continuously differentiable functions $\gamma_1(\tau),\dots,\gamma_r(\tau)$ on $[T_0,\infty)$ with derivatives $\gamma_j'(\tau)=\hat\gamma_j(\tau)(1+o(1))$, where $\hat\gamma_1(\tau),\dots,\hat\gamma_r(\tau)$ are monotonic and are compared in the sense that, for every subset $J\subset\{1,\dots,r\}$, $\#J\geqslant 2$, there exists $j_0=j_0(J)$ such that $\hat\gamma_j(\tau)=o(\hat\gamma_{j_0}(\tau))$ for $j\in J$, $j\neq j_0$, and $\gamma_j(2\tau)\max_{\tau\leqslant u\leqslant 2\tau}1/\hat\gamma_j(u)\ll\tau$, $j=1,\dots,r$, as $\tau\to\infty$.

Theorem 3. *Suppose that the sequences* $\mathfrak{a}_1,\dots,\mathfrak{a}_r$ *are multiplicative, and* $(\gamma_1(\tau),\dots,\gamma_r(\tau))\in U_r(T_0)$. *For* $j=1,\dots,r$, *let* $K_j\in\mathcal{K}$ *and* $f_j(s)\in H_0(K_j)$. *Then, for every* $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T-T_0}\,\mathrm{meas}\Big\{\tau\in[T_0,T]\colon \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\big|\zeta\big(s+\mathrm{i}\gamma_j(\tau);\mathfrak{a}_j\big)-f_j(s)\big|<\varepsilon\Big\}>0.$$

Moreover, the limit

$$\lim_{T\to\infty}\frac{1}{T-T_0}\,\mathrm{meas}\Big\{\tau\in[T_0,T]\colon \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\big|\zeta\big(s+\mathrm{i}\gamma_j(\tau);\mathfrak{a}_j\big)-f_j(s)\big|<\varepsilon\Big\}>0$$

exists for all but at most countably many $\varepsilon>0$.

For example, we may take $\underline{a}=(\sqrt{2},\sqrt{3},\sqrt{5},\dots,\sqrt{p_r})$, where p_r is the rth prime number, and $\gamma(\tau)=\tau\log\tau$, $\tau\geqslant 2$, in Theorem 2, and $\gamma_1(\tau)=\tau\log\tau$, $\gamma_2=\tau^2\log\tau$, $\dots,\gamma_r(\tau)=\tau^r\log\tau$ in Theorem 3.

Similar results can be obtained for more general zeta-functions with Euler product, for example, for the Matsumoto zeta-functions.

For the proof of Theorems 2 and 3, we will apply the probabilistic approach based on limit theorems for probability measures in the space of analytic functions. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, by $H(D)$ the space of analytic functions on $D=\{s\in\mathbb{C}\colon 1/2<\sigma<1\}$ endowed with the topology of uniform convergence on compacta, let, for brevity, $\underline{\mathfrak{a}}=(\mathfrak{a}_1,\dots,\mathfrak{a}_r)$, $\underline{a}=(a_1,\dots,a_r)$, $\underline{\gamma}(\tau)=(\gamma_1(\tau),\dots,\gamma_r(\tau))$, and

$$\underline{\zeta}(s;\underline{\mathfrak{a}})=\big(\zeta(s;\mathfrak{a}_1),\dots,\zeta(s;\mathfrak{a}_r)\big).$$

More precisely, we will consider the weak convergence for

$$P_T^1(A)\stackrel{\mathrm{def}}{=}\frac{1}{T-T_0}\,\mathrm{meas}\big\{\tau\in[T_0,T]\colon \underline{\zeta}\big(s+\mathrm{i}\underline{a}\gamma(\tau);\underline{\mathfrak{a}}\big)\in A\big\},\quad A\in\mathcal{B}\big(H^r(D)\big),$$

and

$$P_T^r(A)\stackrel{\mathrm{def}}{=}\frac{1}{T-T_0}\,\mathrm{meas}\big\{\tau\in[T_0,T]\colon \underline{\zeta}\big(s+\mathrm{i}\underline{\gamma}(\tau);\underline{\mathfrak{a}}\big)\in A\big\},\quad A\in\mathcal{B}\big(H^r(D)\big),$$

as $T\to\infty$.

2 Limit theorems on the torus

Let $\gamma = \{s \in \mathbb{C}\colon |s| = 1\}$ be the unit circle, $\mathbb{P}$ denote the set of all prime numbers, and

$$\Omega = \prod_{p\in\mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. With the product topology and pointwise multiplication, the torus Ω is a compact topological group, therefore on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure exists. For the proof of Theorem 1 in [18], a limit theorem for probability measures on $(\Omega, \mathcal{B}(\Omega))$ was applied. In our case, the above theorem is not sufficient. Define,

$$\underline{\Omega}^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \ldots, r$. Then, again, $\underline{\Omega}^r$ is a compact topological group, therefore, on $(\underline{\Omega}^r, \mathcal{B}(\underline{\Omega}^r))$, the probability Haar measure m_H^r can be defined. This gives the probability space $(\underline{\Omega}^r, \mathcal{B}(\underline{\Omega}^r), m_H^r)$. For $A \in \mathcal{B}(\underline{\Omega}^r)$, define

$$Q_T^1(A) = \frac{1}{T-T_0} \operatorname{meas}\bigl\{\tau \in [T_0, T]\colon \bigl(p^{-\mathrm{i}a_1\gamma(\tau)}\colon p \in \mathbb{P}\bigr), \ldots, \bigl(p^{-\mathrm{i}a_r\gamma(\tau)}\colon p \in \mathbb{P}\bigr) \in A\bigr\}.$$

Lemma 1. *Suppose that $\underline{a}$ and $\gamma(\tau)$ satisfy the hypotheses of Theorem 2. Then Q_T^1 converges weakly to the Haar measure m_H^r as $T \to \infty$.*

Proof. The dual group of $\underline{\Omega}^r$ is isomorphic to

$$\bigoplus_{j=1}^{r} \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_{jp},$$

where $\mathbb{Z}_{jp} = \mathbb{Z}$ for all $j = 1, \ldots, r$ and $p \in \mathbb{P}$. Therefore, the Fourier transform $g_T^1(\underline{k})$ of Q_T^1, $\underline{k} = (\underline{k}_1, \ldots, \underline{k}_r)$, $\underline{k}_j = \{k_{jp} \in \mathbb{Z}\colon p \in \mathbb{P}\}$, is of the form

$$g_T^1(\underline{k}) = \int_{\underline{\Omega}^r} \prod_{j=1}^{r} {\prod_{p\in\mathbb{P}}}^{*} \omega_j^{k_{jp}}(p)\, \mathrm{d}Q_T^1,$$

where $\omega_j(p)$ is the pth component of an element $\omega_j \in \Omega_j$, $p \in \mathbb{P}$, and the star "$*$" shows that only a finite number of integers k_{jp} are distinct from zero. Hence, by the definition of Q_T^1,

$$\begin{aligned} g_T^1(\underline{k}) &= \frac{1}{T-T_0} \int_{T_0}^{T} \prod_{j=1}^{r} {\prod_{p\in\mathbb{P}}}^{*} p^{-\mathrm{i}a_j\gamma(\tau)k_{jp}}\, \mathrm{d}\tau \\ &= \frac{1}{T-T_0} \int_{T_0}^{T} \exp\Biggl\{-\mathrm{i}\gamma(\tau) \sum_{j=1}^{r} a_j {\sum_{p\in\mathbb{P}}}^{*} k_{jp} \log p\Biggr\}\, \mathrm{d}\tau. \end{aligned} \tag{2}$$

Clearly,

$$g_T^1\big((\underline{0},\dots,\underline{0})\big)=1. \tag{3}$$

Now, suppose that $\underline{k}\neq(\underline{0},\dots,\underline{0})$. We have

$$A_{\underline{k}}\stackrel{\mathrm{def}}{=}\sum_{j=1}^{r}a_j{\sum_{p\in\mathbb{P}}}^{*}k_{jp}\log p={\sum_{p\in\mathbb{P}}}^{*}\log p\sum_{j=1}^{r}a_jk_{jp}.$$

Let

$$p_{\min}=\min_{1\leqslant j\leqslant r}\min_{p}\big\{p\colon\ k_{jp}\in\underline{k}_j,\ k_{jp}\neq 0\big\}$$

and

$$p_{\max}=\max_{1\leqslant j\leqslant r}\max_{p}\big\{p\colon\ k_{jp}\in\underline{k}_j,\ k_{jp}\neq 0\big\}.$$

Then there exists at least one $p\in[p_{\min},p_{\max}]$ such that $k_{jp}\neq 0$ for some j, thus, by the linear independence of the numbers $a_1,\dots,a_r$,

$$\beta_p\stackrel{\mathrm{def}}{=}\sum_{j=1}^{r}a_jk_{jp}\neq 0.$$

The numbers β_p are algebraic, moreover, it is well known that the set $\{\log p\colon\ p\in\mathbb{P}\}$ is linearly independent over $\mathbb{Q}$. Therefore, by the Baker theorem, see, for example, [3], the form

$$A_{\underline{k}}={\sum_{p\in\mathbb{P}}}^{*}\beta_p\log p\neq 0.$$

Using the monotonicity of $\gamma'(\tau)$ and the mean value theorem, we find by (2)

$$g_T^1(\underline{k})\ll\frac{1}{|A(\underline{k})|T}\max\biggl(\frac{1}{\gamma'(T)},\frac{1}{\gamma'(T_0)}\biggr). \tag{4}$$

Since $\gamma(\tau)\in U_1(T_0)$, we have $1/\gamma'(T)=o(T)$. This, together with (3) and (4), shows that

$$\lim_{T\to\infty}g_T^1(\underline{k})=\begin{cases}1 & \text{if } \underline{k}=(\underline{0},\dots,\underline{0}),\\ 0 & \text{if } \underline{k}\neq(\underline{0},\dots,\underline{0}).\end{cases}$$

Since the right-hand side of the above equality is the Fourier transform of the Haar measure m_H^r, the lemma is proved. □

For $A\in\mathcal{B}(\underline{\Omega}^r)$, define

$$Q_T^r(A)=\frac{1}{T-T_0}\operatorname{meas}\big\{\tau\in[T_0,T]\colon\ \underline{\zeta}\big(s+\mathrm{i}\underline{\gamma}(\tau);\underline{\mathfrak{a}}\big)\in A\big\}.$$

Lemma 2. *Suppose that* $(\gamma_1(\tau),\dots,\gamma_r(\tau))\in U_r(T_0)$. *Then* Q_T^r *converges weakly to the Haar measure* m_H^r *as* $T\to\infty$.

Proof. As in the proof of Lemma 1, we consider the Fourier transform of Q_T^r

$$g_T^r(\underline{k}) = \frac{1}{T-T_0} \int_{T_0}^{T} \exp\Biggl\{ -\mathrm{i} \sum_{j=1}^{r} \gamma_j(\tau) \sum_{p\in\mathbb{P}}^{*} k_{jp} \log p \Biggr\} \,\mathrm{d}\tau. \tag{5}$$

Obviously,

$$g_T^r\bigl((\underline{0},\dots,\underline{0})\bigr) = 1. \tag{6}$$

Therefore, it remains to consider the case $\underline{k} \neq (\underline{0},\dots,\underline{0})$. For brevity, let

$$b_j = \sum_{p\in\mathbb{P}}^{*} k_{jp} \log p.$$

Since, the set $\{\log p\colon p\in\mathbb{P}\}$ is linearly independent over $\mathbb{Q}$, we have $b_j \neq 0$ for $\underline{k}_j \neq \underline{0}$, $j = 1,\dots,r$. Put

$$A(\tau) = \sum_{j=1}^{r} b_j \gamma_j(\tau).$$

Suppose that $\underline{k}_j \neq \underline{0}$ for $j \in J \subset \{1,\dots,r\}$, $\#J \geqslant 2$. Then there exists $j_0 \in J$ such that $\hat{\gamma}_j(\tau) = o(\hat{\gamma}_{j_0}(\tau))$, $\tau\to\infty$, for $j \in J\setminus\{j_0\}$. Therefore,

$$A'(\tau) = \sum_{j\in J} b_j \gamma_j'(\tau) = \sum_{j\in J} b_j \hat{\gamma}_j(\tau)\bigl(1+o(1)\bigr) = b_{j_0}\hat{\gamma}_{j_0}(\tau)\bigl(1+o(1)\bigr),$$

$$\bigl(A'(\tau)\bigr)^{-1} = \frac{1}{b_{j_0}\hat{\gamma}_{j_0}(\tau)(1+o(1))} = \frac{1}{b_{j_0}\hat{\gamma}_{j_0}(\tau)}\bigl(1+o(1)\bigr)$$

and

$$\frac{1}{b_{j_0}\hat{\gamma}_{j_0}(\tau)} = \frac{(A(\tau))^{-1}}{(1+o(1))} = \bigl(A(\tau)\bigr)^{-1}\bigl(1+o(1)\bigr)$$

as $\tau\to\infty$. Hence, using the monotonicity of $\hat{\gamma}_{j_0}(\tau)$ and the second mean value theorem, we find

$$\begin{aligned}
\int_{T_0}^{T} \cos A(\tau)\,\mathrm{d}\tau &= \int_{\log T}^{T} \cos A(\tau)\,\mathrm{d}\tau + O(\log T) \\
&= \int_{\log T}^{T} \frac{1}{A'(\tau)} \cos A(\tau)\,\mathrm{d}A(\tau) + O(\log T) \\
&= \int_{\log T}^{T} \frac{1}{b_{j_0}\hat{\gamma}_{j_0}(\tau)} \cos A(\tau)\,\mathrm{d}A(\tau) \\
&\quad + \int_{\log T}^{T} \frac{o(1)}{b_{j_0}\hat{\gamma}_{j_0}(\tau)} \cos A(\tau)\,\mathrm{d}A(\tau) + O(\log T)
\end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{\log T}^{T} \frac{1}{b_{j_0}\hat{\gamma}_{j_0}(\tau)} \,\mathrm{d}\bigl(\sin A(\tau)\bigr) \\
&\quad + \int\limits_{\log T}^{T} \frac{o(1)(1+o(1))}{A'(\tau)} \cos A(\tau)\,\mathrm{d}A(\tau) + O(\log T) \\
&= o(T) + \int\limits_{\log T}^{T} o(1)\cos A(\tau)\,\mathrm{d}\tau + O(\log T) \\
&= o(T), \quad T \to \infty,
\end{aligned}
$$

because $1/(\hat{\gamma}_0(\tau)) = o(\tau)$ as $\tau \to \infty$. By the same lines, we obtain

$$
\int\limits_{T_0}^{T} \sin A(\tau)\,\mathrm{d}\tau = o(T).
$$

This, (6) and (5) show that, for $\underline{k} \neq (\underline{0}, \dots, \underline{0})$,

$$
\lim_{T\to\infty} g_T^r(\underline{k}) = 0,
$$

and the lemma follows from (6) in the same way as Lemma 1, because, in the case $\#J = 1$, $A(\tau) = b_j\gamma_j(\tau)$ for some j. □

3 Case of absolutely convergent series

Lemmas 1 and 2 allow to prove limit theorems in the space $H^r(D)$ for measures defined by means of absolutely convergent Dirichlet series.

For fixed $\theta > 1/2$, and $m, n \in \mathbb{N}$, let $v_n(m) = \exp\{-(m/n)^\theta\}$. Define the series

$$
\zeta_n(s; \mathfrak{a}_j) = \sum_{m=1}^{\infty} \frac{a_{jm} v_n(m)}{m^s}, \quad j = 1, \dots, r.
$$

Then, in view of the definition of $v_n(m)$, the latter series are absolutely convergent for $\sigma > 1/2$ [20]. For brevity, let

$$
\underline{\zeta}_n(s; \underline{\mathfrak{a}}) = \bigl(\zeta_n(s; \mathfrak{a}_1), \dots, \zeta_n(s; \mathfrak{a}_r)\bigr)
$$

and, for $\mathcal{B}(H^r(D))$,

$$
P_{T,n}^1(A) = \frac{1}{T - T_0} \operatorname{meas}\bigl\{\tau \in [T_0, T]\colon \underline{\zeta}_n\bigl(s + \mathrm{i}\underline{a}\gamma(\tau); \underline{\mathfrak{a}}\bigr) \in A\bigr\}
$$

and

$$
P_{T,n}^r(A) = \frac{1}{T - T_0} \operatorname{meas}\bigl\{\tau \in [T_0, T]\colon \underline{\zeta}_n\bigl(s + \mathrm{i}\underline{\gamma}(\tau); \underline{\mathfrak{a}}\bigr) \in A\bigr\}.
$$

Denote by $\underline{\omega} = (\omega_1, \ldots, \omega_r)$, $\omega_j \in \Omega_j$, $j = 1, \ldots, r$, the elements of $\underline{\Omega}^r$. Together with series $\zeta_n(s; \mathfrak{a}_j)$, we consider the series

$$\zeta_n(s, \omega_j; \mathfrak{a}_j) = \sum_{m=1}^{\infty} \frac{a_{jm}\omega_j(m)v_n(m)}{m^s}, \quad j = 1, \ldots, r,$$

that are absolutely convergent for $\sigma > 1/2$ as well. Here, for $m \in \mathbb{N}$,

$$\omega_j(m) = \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} \omega_j^l(p), \quad j = 1, \ldots, r.$$

Analogically, let, for $\underline{\omega} \in \underline{\Omega}^r$,

$$\underline{\zeta}_n(s, \underline{\omega}; \underline{\mathfrak{a}}) = \big(\zeta_n(s, \omega_1; \mathfrak{a}_1), \ldots, \zeta_n(s, \omega_r; \mathfrak{a}_r)\big)$$

and, for $\mathcal{B}(H^r(D))$,

$$P^1_{T,n,\underline{\omega}}(A) = \frac{1}{T - T_0} \operatorname{meas}\big\{\tau \in [T_0, T]\colon \underline{\zeta}_n\big(s + \mathrm{i}\underline{a}\gamma(\tau), \underline{\omega}; \underline{\mathfrak{a}}\big) \in A\big\}$$

and

$$P^r_{T,n,\underline{\omega}}(A) = \frac{1}{T - T_0} \operatorname{meas}\big\{\tau \in [T_0, T]\colon \underline{\zeta}_n\big(s + \mathrm{i}\underline{\gamma}(\tau), \underline{\omega}; \underline{\mathfrak{a}}\big) \in A\big\}.$$

Let the mapping $u_n : \underline{\Omega}^r \to H^r(D)$ be given by the formula

$$u_n(\underline{\omega}) = \underline{\zeta}_n(s, \underline{\omega}; \underline{\mathfrak{a}}).$$

Then the mapping u_n is continuous because of the absolute convergence of the series $\zeta_n(s, \omega_j; \mathfrak{a}_j)$. Therefore, the definitions of $P^1_{T,n}$, $P^1_{T,n,\underline{\omega}}$ and Q^1_T, and $P^r_{T,n}$, $P^r_{T,n,\underline{\omega}}$ and Q^r_T, Lemmas 1 and 2, and properties of weak convergence of probability measures [4, Thm. 5.1] lead to the following limit theorems on $(H^r(D), \mathcal{B}(H^r(D)))$.

Lemma 3. *Suppose that* $\underline{a}$ *and* $\gamma(\tau)$ *satisfy the hypotheses of Theorem* 2. *Then* $P^1_{T,n}$ *and* $P^1_{T,n,\underline{\omega}}$ *converge weakly to the measure* $m^r_H u_n^{-1}$ *as* $T \to \infty$.

Lemma 4. *Suppose that* $(\gamma_1(\tau), \ldots, \gamma_r(\tau)) \in U_r(T_0)$. *Then* $P^r_{T,n}$ *and* $P^r_{T,n,\underline{\omega}}$ *converge weakly to the measure* $m^r_H u_n^{-1}$ *as* $T \to \infty$.

4 Mean square estimates

To pass from weak convergence for $P^1_{T,n}$ and $P^r_{T,n}$ to for P^1_T and P^r_T, respectively, as $T \to \infty$, a certain approximation of $\underline{\zeta}(s; \underline{\mathfrak{a}})$ by $\underline{\zeta}_n(s; \underline{\mathfrak{a}})$ is needed. This approximation is based on the mean square estimates for $\zeta(s, \mathfrak{a}_j)$.

Thus, let $\mathfrak{a}$ be an arbitrary periodic sequence of complex numbers, and $a \in \mathbb{R} \setminus \{0\}$.

Lemma 5. *Suppose that* $\gamma(\tau) \in U_1(T_0)$. *Then, for every fixed* σ, $1/2 < \sigma < 1$, *and* $t \in \mathbb{R}$,

$$\int_{T_0}^{T} \left|\zeta\left(\sigma + \mathrm{i}a\gamma(\tau) + \mathrm{i}t; \mathfrak{a}\right)\right|^2 \mathrm{d}\tau \ll_{\sigma,\mathfrak{a}} T\left(1 + |t|\right).$$

Proof. It is well known that, for fixed σ, $1/2 < \sigma < 1$, the Hurwitz zeta-function $\zeta(s, \alpha)$ satisfies

$$\int_{T_0}^{T} \left|\zeta(\sigma + \mathrm{i}t, \alpha)\right|^2 \mathrm{d}t \ll_{\sigma,\alpha} T.$$

This, together with (1), implies the bound

$$\int_{T_0}^{T} \left|\zeta(\sigma + \mathrm{i}t; \mathfrak{a})\right|^2 \mathrm{d}t \ll_{\sigma,\mathfrak{a}} T.$$

From this it follows

$$\int_{T_0}^{|t|+|a|\gamma(\tau)} \left|\zeta(\sigma + \mathrm{i}u; \mathfrak{a})\right|^2 \mathrm{d}u \ll_{\sigma,\mathfrak{a}} \left(|t| + |a|\gamma(\tau)\right).$$

Therefore, for $X \geqslant T_0$, we have that

$$\begin{aligned}
&\int_{X}^{2X} \left|\zeta\left(\sigma + \mathrm{i}a\gamma(\tau) + \mathrm{i}t; \mathfrak{a}\right)\right|^2 \mathrm{d}\tau \\
&\quad = \frac{1}{a} \int_{X}^{2X} \frac{1}{\gamma'(\tau)} \left|\zeta\left(\sigma + \mathrm{i}a\gamma(\tau) + \mathrm{i}t; \mathfrak{a}\right)\right|^2 \mathrm{d}\gamma(\tau) \\
&\quad \ll_a \max_{X \leqslant \tau \leqslant 2X} \frac{1}{\gamma'(\tau)} \left| \int_{X}^{2X} \mathrm{d}\left(\int_{T_0}^{t+a\gamma(\tau)} \left|\zeta(\sigma + \mathrm{i}u; \mathfrak{a})\right|^2 \mathrm{d}u \right) \right| \\
&\quad \ll_{a,\sigma,\mathfrak{a}} \left(|t| + |a|\gamma(2X)\right) \max_{X \leqslant \tau \leqslant 2X} \frac{1}{\gamma'(\tau)} \ll_{a,\sigma,\mathfrak{a}} X\left(1 + |t|\right)
\end{aligned}$$

because $\gamma(\tau) \in U_1(T_0)$. Taking $T2^{-k-1}$ and summing over $k = 0, 1, \ldots,$ give the estimate of the lemma. □

Lemma 6. *Let* $(\gamma_1(\tau), \ldots, \gamma_r(\tau)) \in U_r(T_0)$. *Then, for every fixed* σ, $1/2 < \sigma < 1$, *and* $t \in \mathbb{R}$,

$$\int_{T_0}^{T} \left|\zeta\left(\sigma + \mathrm{i}\gamma_j(\tau) + \mathrm{i}t; \mathfrak{a}\right)\right|^2 \mathrm{d}\tau \ll_{\sigma} T\left(1 + |t|\right)$$

for $j = 1, \ldots, r$.

Proof. Using the notation of Lemma 5, we have

$$\begin{aligned}
&\int_X^{2X} \left|\zeta\left(\sigma + \mathrm{i}\gamma_j(\tau) + \mathrm{i}t; \mathfrak{a}\right)\right|^2 \mathrm{d}\tau \\
&\quad = \int_X^{2X} \frac{1}{\gamma_j'(\tau)} \left|\zeta\left(\sigma + \mathrm{i}\gamma_j(\tau) + \mathrm{i}t; \mathfrak{a}\right)\right|^2 \mathrm{d}\gamma_j(\tau) \\
&\quad = \int_X^{2X} \frac{(1+o(1))}{\hat{\gamma}_j(\tau)} \mathrm{d}\left(\int_{T_0}^{t+\gamma_j(\tau)} \left|\zeta(\sigma + \mathrm{i}u; \mathfrak{a})\right|^2 \mathrm{d}u\right) \\
&\quad = \int_X^{2X} \frac{1}{\hat{\gamma}_j(\tau)} \mathrm{d}\left(\int_{T_0}^{t+\gamma_j(\tau)} \left|\zeta(\sigma + \mathrm{i}u; \mathfrak{a})\right|^2 \mathrm{d}u\right) \\
&\qquad + \int_X^{2X} \frac{o(1)(1+o(1))}{\gamma_j'(\tau)} \mathrm{d}\left(\int_{T_0}^{t+\gamma_j(\tau)} \left|\zeta(\sigma + \mathrm{i}u; \mathfrak{a})\right|^2 \mathrm{d}u\right) \\
&\quad \ll_{\sigma,\mathfrak{a}} |t| + \gamma_j(2X) \max_{X \leqslant \tau \leqslant 2X} \frac{1}{\hat{\gamma}_j(\tau)} + \int_X^{2X} o(1) \left|\zeta\left(\sigma + \mathrm{i}\gamma_j(\tau) + \mathrm{i}t; \mathfrak{a}\right)\right|^2 \mathrm{d}\tau.
\end{aligned}$$

Hence,

$$\int_X^{2X} \left|\zeta\left(\sigma + \mathrm{i}\gamma_j(\tau) + \mathrm{i}t; \mathfrak{a}\right)\right|^2 \mathrm{d}\tau \ll_{\sigma,\mathfrak{a}} X\left(1 + |t|\right)\left(1 + r(X)\right) \ll_{\sigma,\mathfrak{a}} X\left(1 + |t|\right),$$

where $r(X) \to 0$ as $X \to \infty$. This proves the lemma. □

Lemmas 5 and 6 have their modifications for

$$\zeta(s, \omega; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}, \quad \sigma > 1,$$

with $\omega \in \Omega$. We note that the latter series is uniformly convergent on compact subsets of the strip D for almost all ω with respect to the Haar measure on $(\Omega, \mathcal{B}(\Omega))$.

Lemma 7. *Suppose that $\gamma(\tau) \in U_1(T_0)$. Then, for every fixed σ, $1/2 < \sigma < 1$, and $t \in \mathbb{R}$,*

$$\int_{T_0}^{T} \left|\zeta\left(\sigma + \mathrm{i}a\gamma(\tau) + \mathrm{i}t, \omega; \mathfrak{a}\right)\right|^2 \mathrm{d}\tau \ll_{\sigma,a,\mathfrak{a}} T\left(1 + |t|\right)$$

for almost all $\omega \in \Omega$.

Proof. Since, for almost all $\omega \in \Omega$, see [20],

$$\int_{T_0}^{T} \left|\zeta(\sigma + \mathrm{i}t, \omega; \mathfrak{a})\right|^2 \mathrm{d}t \ll_{\sigma,\mathfrak{a}} T, \tag{7}$$

the proof coincides with that of Lemma 5. □

Lemma 8. *Let* $(\gamma_1(\tau), \dots, \gamma_r(\tau)) \in U_r(T_0)$. *Then, for every fixed* σ, $1/2 < \sigma < 1$, *and* $t \in \mathbb{R}$,

$$\int_{T_0}^{T} \left|\zeta\big(\sigma + \mathrm{i}a\gamma_j(\tau) + \mathrm{i}t, \omega; \mathfrak{a}\big)\right|^2 \mathrm{d}\tau \ll_{\sigma,\mathfrak{a}} T\big(1 + |t|\big)$$

for almost all $\omega \in \Omega$, $j = 1, \dots, r$.

Proof. We repeat the proof of Lemma 6 and apply the estimate (7). □

Now, we will apply Lemmas 5–8 for the approximation of $\underline{\zeta}(s; \underline{\mathfrak{a}})$ by $\underline{\zeta}_n(s; \underline{\mathfrak{a}})$. For $g_1, g_2 \in H(D)$, let

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l\colon l \in \mathbb{N}\} \subset D$ is a sequence of compact sets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$, for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some l. Then ρ is a metric in $H(D)$ inducing its topology of uniform convergence on compacta. Let $\underline{g}_1 = (g_{11}, \dots, g_{1r})$, $\underline{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D)$. Then taking

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1\leqslant j\leqslant r} \rho_j(g_{1j}, g_{2j})$$

gives the metric in the space $H^r(D)$ inducing its product topology.

Lemma 9. *Suppose that* $a_1, \dots, a_r \in \mathbb{R} \setminus \{0\}$ *and* $\gamma(\tau) \in U_1(T_0)$. *Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T - T_0} \int_{T_0}^{T} \underline{\rho}\big(\underline{\zeta}\big(s + \mathrm{i}\underline{a}\gamma(\tau); \underline{\mathfrak{a}}\big), \underline{\zeta}_n\big(s + \mathrm{i}\underline{a}\gamma(\tau); \underline{\mathfrak{a}}\big)\big) \,\mathrm{d}\tau = 0.$$

Moreover, for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T - T_0} \int_{T_0}^{T} \underline{\rho}\big(\underline{\zeta}\big(s + \mathrm{i}\underline{a}\gamma(\tau), \underline{\omega}; \underline{\mathfrak{a}}\big), \underline{\zeta}_n\big(s + \mathrm{i}\underline{a}\gamma(\tau), \underline{\omega}; \underline{\mathfrak{a}}\big)\big) \,\mathrm{d}\tau = 0.$$

Proof. From the definitions of the metrics $\underline{\rho}$ and ρ it follows that it is sufficient to prove that, for every compact set $K \subset D$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-T_0} \int_{T_0}^{T} \sup_{s\in K} \bigl|\zeta\bigl(s+\mathrm{i}a_j\gamma(\tau);\mathfrak{a}_j\bigr) - \zeta_n\bigl(s+\mathrm{i}a_j\gamma(\tau);\mathfrak{a}_j\bigr)\bigr| \,\mathrm{d}\tau = 0$$

for all $j = 1, \dots, r$.

Let $\mathfrak{a}$ and $a \neq 0$ be arbitrary. The definition of $\zeta_n(s;\mathfrak{a})$ and the classical Mellin formula

$$\frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Gamma(s) b^{-s} \,\mathrm{d}s = \mathrm{e}^{-b}, \quad b, c > 0,$$

where $\Gamma(s)$ denotes the Euler gamma-function, yield the integral representation [20]

$$\zeta_n(s;\mathfrak{a}) = \frac{1}{2\pi\mathrm{i}} \int_{\theta-\mathrm{i}\infty}^{\theta+\mathrm{i}\infty} \zeta(s+z;\mathfrak{a}) l_n(z) \frac{\mathrm{d}z}{z}, \quad l_n(s) = \frac{s}{\theta}\Gamma\biggl(\frac{s}{\theta}\biggr) n^s.$$

Therefore, taking $\theta_1 > 0$, we have

$$\zeta_n(s;\mathfrak{a}) - \zeta(s;\mathfrak{a}) = \frac{1}{2\pi\mathrm{i}} \int_{-\theta_1-\mathrm{i}\infty}^{-\theta_1+\mathrm{i}\infty} \zeta(s+z;\mathfrak{a}) l_n(z) \frac{\mathrm{d}z}{z} + R_n(s;\mathfrak{a}), \tag{8}$$

where

$$R_n(s;\mathfrak{a}) = r_{\mathfrak{a}} \frac{l_n(1-s)}{1-s}.$$

Let $K \subset D$ be an arbitrary compact set. Denote by $\sigma + \mathrm{i}v$ the points of K, and fix $\varepsilon > 0$ such that $1/2 + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$. Then, by (8),

$$\begin{aligned} &\bigl|\zeta\bigl(s+\mathrm{i}a\gamma(\tau);\mathfrak{a}\bigr) - \zeta_n\bigl(s+\mathrm{i}a\gamma(\tau);\mathfrak{a}\bigr)\bigr| \\ &\quad \ll \int_{-\infty}^{\infty} \bigl|\zeta\bigl(s+\mathrm{i}a\gamma(\tau) - \theta_1 + \mathrm{i}t;\mathfrak{a}\bigr)\bigr| \frac{|l_n(-\theta_1+\mathrm{i}t)|}{|-\theta_1+\mathrm{i}t|} \,\mathrm{d}t + \bigl|R_n\bigl(s+\mathrm{i}a\gamma(\tau);\mathfrak{a}\bigr)\bigr|. \end{aligned}$$

Thus,

$$\frac{1}{T-T_0} \int_{T_0}^{T} \sup_{s\in K} \bigl|\zeta\bigl(s+\mathrm{i}a\gamma(\tau);\mathfrak{a}\bigr) - \zeta_n v\bigl(s+\mathrm{i}a\gamma(\tau);\mathfrak{a}\bigr)\bigr| \,\mathrm{d}\tau \ll I_1 + I_2, \tag{9}$$

where

$$I_1 = \int_{-\infty}^{\infty} \frac{1}{T-T_0} \int_{T_0}^{T} \biggl(\biggl|\zeta\biggl(\frac{1}{2}+\varepsilon+\mathrm{i}\bigl(t+a\gamma(\tau)\bigr);\mathfrak{a}\biggr)\biggr| \,\mathrm{d}\tau\biggr) \sup_{s\in K} \frac{|l_n(\frac{1}{2}+\varepsilon-s+\mathrm{i}t)|}{|\frac{1}{2}+\varepsilon-s+\mathrm{i}t|} \,\mathrm{d}t$$

and

$$I_2 = \frac{1}{T-T_0}\int\limits_{T_0}^{T} \sup_{s\in K}\big|R_n\big(s+\mathrm{i}a\gamma(\tau);\mathfrak{a}\big)\big|\,\mathrm{d}\tau.$$

Since in the definition of $l_n(s)$ the gamma-function occurs, we can use the estimate

$$\Gamma(\sigma+\mathrm{i}t) \ll \exp\{-c|t|\}, \quad c>0,$$

which is uniform in σ, $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, for arbitrary $\sigma_1 < \sigma_2$. Therefore, for $s\in K$,

$$\begin{aligned}\frac{|l_n(\frac{1}{2}+\varepsilon-s+\mathrm{i}t)|}{|\frac{1}{2}+\varepsilon-s+\mathrm{i}t|} &= \frac{n^{1/2+\varepsilon-\sigma}}{\theta}\left|\Gamma\left(\frac{\frac{1}{2}+\varepsilon-\sigma}{\theta}+\frac{\mathrm{i}(t-v)}{\theta}\right)\right| \\ &\ll_{\theta,K} n^{-\varepsilon}\exp\left\{-\frac{c_1}{\theta}|t|\right\}, \quad c_1>0. \qquad (10)\end{aligned}$$

Similarly, we find

$$R_n\big(s+\mathrm{i}a\gamma(\tau);\mathfrak{a}\big) \ll_{\theta,\mathfrak{a},K} n^{1-\sigma}\exp\left\{-\frac{c_2}{\theta}|a|\gamma(\tau)\right\}, \quad c_2>0. \qquad (11)$$

Now, putting $\theta = 1/2+\varepsilon$, and estimate (10) together with Lemma 5 yield

$$I_1 \ll_{\varepsilon,K,\mathfrak{a}} n^{-\varepsilon}\int\limits_{-\infty}^{\infty}\big(1+|t|\big)\exp\{-c_3|t|\}\,\mathrm{d}t \ll_{\varepsilon,K,\mathfrak{a}} n^{-\varepsilon}, \quad c_3>0. \qquad (12)$$

Moreover, properties of the functions $\gamma(\tau)$ and (11) show that with $c_4>0$

$$\begin{aligned}I_2 &\ll_{\varepsilon,\mathfrak{a},K} n^{1/2-2\varepsilon}\frac{1}{T-T_0}\int\limits_{T_0}^{T}\exp\{-c_4|a|\gamma(\tau)\}\,\mathrm{d}\tau \\ &\ll_{\varepsilon,\mathfrak{a},K} n^{1/2-2\varepsilon}\left(\frac{\log T}{T}+\frac{1}{T}\int\limits_{\log T}^{T}\exp\{-c_4|a|\gamma(\tau)\}\,\mathrm{d}\tau\right) \\ &\ll n^{1/2-2\varepsilon}\left(\frac{\log T}{T}+\frac{1}{T}\exp\left\{-\frac{c_4}{2}|a|\gamma(\log T)\right\}\int\limits_{\log T}^{T}\exp\left\{-\frac{c_4}{2}|a|\gamma(\tau)\right\}\mathrm{d}\tau\right) \\ &= o(T)\end{aligned}$$

as $T\to\infty$. This, (12) and (9) prove the first assertion of the lemma.

For almost all $\omega\in\Omega$, the function $\zeta(s,\omega;\mathfrak{a})$ is analytic in the half-plane $\sigma>1/2$. Therefore, the second assertion of the lemma is obtained similarly to that of the first with using Lemma 7. In this case, we have not the integral I_2. □

Lemma 10. *Suppose that* $(\gamma_1(\tau), \ldots, \gamma_r(\tau)) \in U_r(T_0)$. *Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-T_0} \int_{T_0}^{T} \underline{\rho}\big(\underline{\zeta}\big(s+\mathrm{i}\underline{\gamma}(\tau); \underline{\mathfrak{a}}\big), \underline{\zeta}_n\big(s+\mathrm{i}\underline{\gamma}(\tau); \underline{\mathfrak{a}}\big)\big)\,\mathrm{d}\tau = 0.$$

Moreover, for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-T_0} \int_{T_0}^{T} \underline{\rho}\big(\underline{\zeta}\big(s+\mathrm{i}\underline{\gamma}(\tau), \underline{\omega}; \underline{\mathfrak{a}}\big), \underline{\zeta}_n\big(s+\mathrm{i}\underline{\gamma}(\tau), \underline{\omega}; \underline{\mathfrak{a}}\big)\big)\,\mathrm{d}\tau = 0.$$

Proof. We use Lemmas 6 and 8 and follow the proof of Lemma 9. □

5 Limit theorems for $\underline{\zeta}(s; \mathfrak{a})$

The results of Sections 3 and 4 are sufficient to prove limit theorems for $\underline{\zeta}(s; \mathfrak{a})$ without explicit forms of limit measures. Together with P_T^1 and P_T^r, we will prove the weak convergence, as $T \to \infty$, for

$$P^1_{T,\underline{\omega}}(A) = \frac{1}{T-T_0} \operatorname{meas}\big\{\tau \in [T_0, T]\colon \underline{\zeta}\big(s + \mathrm{i}\underline{a}\gamma(\tau), \underline{\omega}; \underline{\mathfrak{a}}\big) \in A\big\},$$

and

$$P^r_{T,\underline{\omega}}(A) = \frac{1}{T-T_0} \operatorname{meas}\big\{\tau \in [T_0, T]\colon \underline{\zeta}\big(s + \mathrm{i}\underline{\gamma}(\tau), \underline{\omega}; \underline{\mathfrak{a}}\big) \in A\big\},$$

where $A \in \mathcal{B}(H^r(D))$ and $\underline{\omega} \in \underline{\Omega}$.

Theorem 4. *Suppose that* $\underline{a}$ *and* $\gamma(\tau)$ *satisfy hypotheses of Theorem* 2. *Then, on* $(H^r(D), \mathcal{B}(H^r(D)))$, *there exists a probability measure* P^1 *such that* P_T^1 *and* $P^1_{T,\underline{\omega}}$ *both converge weakly to* P^1 *as* $T \to \infty$.

Proof. Let, for brevity, $V_n = m_H^r u_n^{-1}$, where u_n is the mapping from Lemma 3. Using the absolute convergence for the series $\zeta_n(s; \mathfrak{a}_j)$, we obtain by a standard way, see, for example, [14], that the sequence of probability measures $\{V_n\colon n \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H^r(D)$ such that $V_n(K) > 1-\varepsilon$ for all $n \in \mathbb{N}$. Then, by the Prokhorov theorem [4], the sequence $\{V_n\}$ is relatively compact. In what follows, we will use the language of random elements. Let θ_T be a random variable on a certain probability space with measure μ, and uniformly distributed on $[T_0, T]$. Define the $H^r(D)$-valued random element

$$\underline{X}^1_{T,n} = \underline{X}^1_{T,n}(s) = \underline{\zeta}_n\big(s + \mathrm{i}\underline{a}\gamma(\theta_T); \underline{\mathfrak{a}}\big),$$

and denote by $\underline{X}^1_n = \underline{X}^1_n(s)$ the $H^r(D)$-valued random element with the distribution V_n. Then the assertion of Lemma 3 can be written in the form

$$\underline{X}^1_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}^1_n. \tag{13}$$

The relative compactness of $\{V_n\}$ implies the existence of subsequences $\{V_{n_k}\}$ such that V_{n_k} converges weakly to a certain probability measure P^1 on $(H^r(D), \mathcal{B}(H^r(D)))$ as $k \to \infty$. Thus,

$$\underline{X}^1_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P^1. \tag{14}$$

Define one more $H^r(D)$-valued random element

$$\underline{X}^1_T = \underline{X}^1_T(s) = \underline{\zeta}\big(s + \mathrm{i}\underline{a}\gamma(\theta_T); \underline{\mathfrak{a}}\big).$$

Then, by the first assertion of Lemma 9, we find that, for every $\varepsilon > 0$,

$$\begin{aligned}&\lim_{n\to\infty} \limsup_{T\to\infty} \mu\big\{\underline{\rho}\big(\underline{X}^1_T, \underline{X}^1_{T,n}\big) \geqslant \varepsilon\big\}\\ &\quad\leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-T_0} \int_{T_0}^{T} \underline{\rho}\big(\underline{\zeta}\big(s + \mathrm{i}\underline{a}\gamma(\tau), \underline{\mathfrak{a}}\big), \underline{\zeta}_n\big(s + \mathrm{i}\underline{a}\gamma(\tau), \underline{\mathfrak{a}}\big)\big)\, \mathrm{d}\tau = 0.\end{aligned}$$

This, (13) and (14) show that all hypotheses of Theorem 4.2 from [4] are satisfied. Therefore, we have the relation

$$\underline{X}^1_T \xrightarrow[T\to\infty]{\mathcal{D}} P^1, \tag{15}$$

or that P^1_T converges weakly to P^1 as $T \to \infty$. Also, in view of (15), the measure P^1 is independent of the subsequence $\{V_{n_k}\}$. Thus,

$$\underline{X}^1_n \xrightarrow[T\to\infty]{\mathcal{D}} P^1. \tag{16}$$

To obtain the weak convergence for $P^1_{T,\underline{\omega}}$, introduce the $H^r(D)$-valued random elements

$$\underline{X}^1_{T,n,\underline{\omega}} = \underline{X}^1_{T,n,\underline{\omega}}(s) = \underline{\zeta}_n\big(s + \mathrm{i}\underline{a}\gamma(\theta_T), \underline{\omega}; \underline{\mathfrak{a}}\big)$$

and

$$\underline{X}^1_{T,\underline{\omega}} = \underline{X}^1_{T,\underline{\omega}}(s) = \underline{\zeta}\big(s + \mathrm{i}\underline{a}\gamma(\theta_T), \underline{\omega}; \underline{\mathfrak{a}}\big).$$

Then, repeating the above arguments for $\underline{X}^1_{T,n,\underline{\omega}}$ and $\underline{X}^1_{T,\underline{\omega}}$ (all relations are true for almost all $\underline{\omega} \in \underline{\Omega}^r$) and using (16), we obtain the weak convergence of $P^1_{T,\underline{\omega}}$ to P^1 as $T \to \infty$. The theorem is proved. □

Theorem 5. *Suppose that $(\gamma_1(\tau), \dots, \gamma_r(\tau)) \in U_r(T_0)$. Then, on $(H^r(D), \mathcal{B}(H^r(D)))$, there exists a probability measure P^r such that P^r_T and $P^r_{T,\underline{\omega}}$ both converge weakly to P^r as $T \to \infty$.*

Proof. We use arguments similar to those of the proof of Theorem 4 with application of Lemmas 4 and 10. □

6 Identification of the limit measures

In this section, we identify the measures P^1 and P^r in Theorems 4 and 5. For this, we will use some results of ergodic theory.

For brevity, let, for $\tau \geqslant T_0$,

$$\underline{a}_\tau^1 = \{ (p^{-\mathrm{i}a_1\gamma(\tau)}\colon p \in \mathbb{P}), \dots, (p^{-\mathrm{i}a_r\gamma(\tau)}\colon p \in \mathbb{P}) \}$$

and

$$\underline{a}_\tau^r = \{ (p^{-\mathrm{i}\gamma_1(\tau)}\colon p \in \mathbb{P}), \dots, (p^{-\mathrm{i}\gamma_r(\tau)}\colon p \in \mathbb{P}) \}.$$

Clearly, $\underline{a}_\tau^1, \underline{a}_\tau^r \in \underline{\Omega}^r$. On $\underline{\Omega}^r$, define the families of transformations $\{\Phi_\tau^1\colon \tau \geqslant T_0\}$ and $\{\Phi_\tau^r\colon \tau \geqslant T_0\}$, where

$$\Phi_\tau^1(\underline{\omega}) = \underline{a}_\tau^1 \underline{\omega} \quad \text{and} \quad \Phi_\tau^r(\underline{\omega}) = \underline{a}_\tau^r \underline{\omega}, \quad \underline{\omega} \in \underline{\Omega}^r.$$

Then $\{\Phi_\tau^1\}$ and $\{\Phi_\tau^r\}$ are families of measurable measure preserving (because of invariance of the Haar measure m_H^r) transformations on $\underline{\Omega}^r$. Recall that a set $A \in \mathcal{B}(\underline{\Omega}^r)$ is called invariant with respect to $\{\Phi_\tau^k\colon \tau \geqslant T_0\}$ if, for every $\tau \geqslant T_0$, the sets A and $A_\tau = \Phi_\tau^k(A)$ can differ one from other at most by a set of m_H^r-measure zero, $k = 1$ or $k = r$. All invariant sets forms a σ-field. The family $\{\Phi_\tau^k\}$ is called ergodic if its σ-field of invariant sets consists only from sets of m_H^r-measure zero or one.

Lemma 11. *The families $\{\Phi_\tau^1\}$ and $\{\Phi_\tau^r\}$ are ergodic.*

Proof. We consider only $\{\Phi_\tau^1\}$ because the case $\{\Phi_\tau^r\}$ is similar, and apply the Fourier transform method. In the proof of Lemma 1, we already have used that the characters χ of the group $\underline{\Omega}^r$ are of the form

$$\chi(\underline{\omega}) = \prod_{j=1}^r \prod_{p \in \mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p).$$

Thus, if the character χ is nontrivial ($\chi(\underline{\omega}) \not\equiv 1$), we have

$$\chi(\underline{a}_\tau^1) = \prod_{j=1}^r \prod_{p \in \mathbb{P}}{}^{*} p^{-\mathrm{i}a_j k_{jp}\gamma(\tau)} = \exp\Biggl\{ -\mathrm{i}\gamma(\tau) \sum_{j=1}^r a_j \sum_{p \in \mathbb{P}}{}^{*} k_{jp} \log p \Biggr\}.$$

Since the character χ is nontrivial, $\underline{k} \neq (\underline{0}, \dots, \underline{0})$. Thus, in the proof of Lemma 1, we have seen that

$$\sum_{j=1}^r a_j \sum_{p \in \mathbb{P}}{}^{*} k_{jp} \log p \neq 0.$$

Therefore, there exists a value $\tau_0 \geqslant T_0$ such that

$$\chi(\underline{a}_{\tau_0}^1) \neq 1. \tag{17}$$

Now, let A be a invariant set with respect to $\{\Phi^1_\tau\}$, and let I_A is its indicator function. Then, for almost all $\underline{\omega} \in \underline{\Omega}^r$,

$$I_A\big(\underline{a}^1_\tau \underline{\omega}\big) = I_A(\underline{\omega}).$$

Thus, in view of the invariance of m^r_H, the Fourier transform $\hat{I}_A(\chi)$ is

$$\begin{aligned}\hat{I}_A(\chi) &= \int\limits_{\underline{\Omega}^r} \chi(\underline{\omega}) I_A(\underline{\omega})\,\mathrm{d}m^r_H = \int\limits_{\underline{\Omega}^r} \chi\big(\underline{a}^1_{\tau_0}\underline{\omega}\big) I_A\big(\underline{a}^1_{\tau_0}\underline{\omega}\big)\,\mathrm{d}m^r_H \\ &= \chi\big(\underline{a}^1_{\tau_0}\big) \int\limits_{\underline{\Omega}^r} \chi(\underline{\omega}) I_A(\underline{\omega})\,\mathrm{d}m^r_H = \chi\big(\underline{a}^1_{\tau_0}\big)\hat{I}_A(\chi).\end{aligned}$$

Therefore, taking into account (17), we have

$$\hat{I}_A(\chi) = 0 \tag{18}$$

for all nontrivial characters of $\underline{\Omega}^r$.

Denote by χ_0 the trivial character of $\underline{\Omega}^r$, and suppose that $\hat{I}(\chi_0) = c$. Then using the orthogonality of characters and (18) give the equality

$$\hat{I}_A(\chi) = c \int\limits_{\underline{\Omega}^r} \chi(\underline{\omega})\,\mathrm{d}m^r_H = c\hat{1}(\chi) = \hat{c}(\chi)$$

for every character χ of $\underline{\Omega}^r$. This shows that $I_A(\underline{\omega}) = c$ for almost all $\underline{\omega} \in \underline{\Omega}^r$. Since $c = 0$ or $c = 1$, we obtain that $m^r_H(A) = 0$ or $m^r_H(A) = 1$. The lemma is proved. □

Lemma 11 allows to identify the limit measures in Theorems 4 and 5. On the probability space $(\underline{\Omega}^r, \mathcal{B}(\underline{\Omega}^r), m^r_H)$, define the $H^r(D)$-valued random element

$$\underline{\zeta}(s, \underline{\omega}; \underline{\mathfrak{a}}) = \big(\zeta(s, \omega_1; \mathfrak{a}_1), \ldots, \zeta(s, \omega_r; \mathfrak{a}_r)\big),$$

where

$$\zeta(s, \omega_j; \mathfrak{a}_j) = \sum_{m=1}^{\infty} \frac{a_{jm}\omega_j(m)}{m^s}, \quad j = 1, \ldots, r.$$

We note that the latter series, for almost all ω_j, are uniformly convergent on compact subsets of D. Moreover, in view of multiplicativity of a_{jm}, for almost all ω_j, the equality

$$\zeta(s, \omega_j; \mathfrak{a}_j) = \prod_{p \in \mathbb{P}} \left(1 + \sum_{k=1}^{\infty} \frac{a_{jp^k}\omega_j^k(p)}{p^{ks}}\right)$$

holds. Let $P_{\underline{\zeta}}$ be the distribution of the random element $\underline{\zeta}(s, \underline{\omega}; \underline{\mathfrak{a}})$, i.e.,

$$P_{\underline{\zeta}}(A) = m^r_H\big\{\underline{\omega} \in \underline{\Omega}^r\colon\ \underline{\zeta}(s, \underline{\omega}; \underline{\mathfrak{a}}) \in A\big\}, \quad A \in \mathcal{B}\big(H^r(D)\big).$$

Theorem 6. *Under hypotheses of Theorems* 2 *and* 3*, P_T^1 and P_T^r converge weakly to the measure $P_{\underline{\zeta}}$ as $T \to \infty$.*

Proof. In view of Theorems 4 and 5, it suffices to prove that P^1 and P^r coincides with $P_{\underline{\zeta}}$. We consider only the case of P^1.

Let A be a fixed continuity set of the measure P^1, i.e., $P^1(\partial A) = 0$, where ∂A is the boundary of A. Then the equivalent of weak convergence of probability measures in terms of continuity sets [4] and Theorem 4 imply

$$\lim_{T\to\infty} \frac{1}{T-T_0} \operatorname{meas}\bigl\{\tau \in [T_0,T]\colon \underline{\zeta}\bigl(s+\mathrm{i}\underline{a}\gamma(\tau), \underline{\omega}; \underline{\mathfrak{a}}\bigr) \in A\bigr\} = P^1(A). \tag{19}$$

On the probability space $(\underline{\Omega}^r, \mathcal{B}(\underline{\Omega}^r), m_H^r)$, define the random variable

$$\theta(\underline{\omega}) = \begin{cases} 1 & \text{if } \underline{\zeta}(s,\underline{\omega};\underline{\mathfrak{a}}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the expectation of $\theta(\underline{\omega})$ is

$$\mathbf{E}\theta = \int\limits_{\underline{\Omega}^r} \theta \,\mathrm{d}m_H^r = m_H^r\bigl\{\underline{\omega} \in \underline{\Omega}^r\colon \underline{\zeta}(s,\underline{\omega};\underline{\mathfrak{a}}) \in A\bigr\} = P_{\underline{\zeta}}(A). \tag{20}$$

In view of Lemma 11, the random process $\theta(\Phi_\tau^1(\underline{\omega}))$ is ergodic. Therefore, by the Birkhoff–Khintchine ergodic theorem [5], for almost all $\underline{\omega} \in \underline{\Omega}^r$,

$$\lim_{T\to\infty} \frac{1}{T-T_0} \int\limits_{T_0}^{T} \theta\bigl(\Phi_\tau^1(\underline{\omega})\bigr)\,\mathrm{d}\tau = \mathbf{E}\theta. \tag{21}$$

On the other hand, by the definitions of θ and Φ_τ^1,

$$\frac{1}{T-T_0} \int\limits_{T_0}^{T} \theta\bigl(\Phi_\tau^1(\underline{\omega})\bigr)\,\mathrm{d}\tau = \frac{1}{T-T_0} \operatorname{meas}\bigl\{\tau \in [T_0,T]\colon \underline{\zeta}\bigl(s+\mathrm{i}\underline{a}\gamma(\tau), \underline{\omega}; \underline{\mathfrak{a}}\bigr) \in A\bigr\}.$$

Thus, in virtue of (20) and (21),

$$\lim_{T\to\infty} \frac{1}{T-T_0} \operatorname{meas}\bigl\{\tau \in [T_0,T]\colon \underline{\zeta}\bigl(s+\mathrm{i}\underline{a}\gamma(\tau), \underline{\omega}; \underline{\mathfrak{a}}\bigr) \in A\bigr\} = P_{\underline{\zeta}}(A).$$

This, together with (19), implies the equality $P^1(A) = P_{\underline{\zeta}}(A)$ for all continuity sets A of P^1. Hence, $P^1(A) = P_{\underline{\zeta}}(A)$ for all $A \in \mathcal{B}(H^r(D))$. The theorem is proved. □

7 Support

For the proof of universality theorems, supports of limit measures in the space of analytic functions play the crucial role. Recall that the support of a probability measure P on

$(\mathbb{X},\mathcal{B}(\mathbb{X}))$ is a minimal closed set S_P such that $P(S_P)=1$. The set S_P consists of all elements $x\in\mathbb{X}$ such that, for every open neighbourhood G of x, the inequality $P(G)>0$ is satisfied.

Theorem 7. *Suppose that the sequences $\mathfrak{a}_1,\dots,\mathfrak{a}_r$ are multiplicative. Then the support of the measure $P_{\underline{\zeta}}$ is the set*

$$\bigl(\bigl\{g\in H(D)\colon\ g(s)\neq 0 \text{ or } g(s)\equiv 0\bigr\}\bigr)^r.$$

Proof. Denote by m_{jH} the probability Haar measure on $(\varOmega_j,\mathcal{B}(\varOmega_j))$. Then m_H^r is the product of the measures $m_{1H},\dots,m_{rH}$, i.e., for $A=A_1\times\dots\times A_r\in\mathcal{B}(H^r(D))$ with $A_j\in\mathcal{B}(H(D))$,

$$m_H^r(A)=m_{1H}(A_1)\cdots m_{rH}(A_r).$$

The space $H^r(D)$ is separable, therefore [4],

$$\mathcal{B}\bigl(H^r(D)\bigr)=\underbrace{\mathcal{B}\bigl(H(D)\bigr)\times\dots\times\mathcal{B}\bigl(H(D)\bigr)}_{r}.$$

Thus, it suffices to consider $P_{\underline{\zeta}}$ on the sets $A=A_1\times\dots\times A_r$, $A_1,\dots,A_r\in\mathcal{B}(H(D))$. It is known [20] that the supports of the measures

$$P_{\zeta_j}(A)=m_{jH}\bigl\{\omega_j\in\varOmega_j\colon\ \zeta(s,\omega_j;\mathfrak{a}_j)\in A_j\bigr\},\quad A_j\in\mathcal{B}\bigl(H(D)\bigr),\ j=1,\dots,r,$$

is the set $\{g\in H(D)\colon\ g(s)\neq 0$ or $g(s)\equiv 0\}$. Moreover, by the above remarks,

$$\begin{aligned}P_{\underline{\zeta}}(A)&=m_{jH}\bigl\{\underline{\omega}\in\underline{\varOmega}^r\colon\ \underline{\zeta}(s,\underline{\omega};\underline{\mathfrak{a}})\in A\bigr\}\\&=m_{1H}\bigl\{\omega_1\in\varOmega_1\colon\ \zeta(s,\omega_1;\mathfrak{a}_1)\in A_1\bigr\}\cdots m_{rH}\bigl\{\omega_r\in\varOmega_r\colon\ \zeta(s,\omega_r;\mathfrak{a}_r)\in A_r\bigr\}\\&=P_{\zeta_1}(A_1)\cdots P_{\zeta_r}(A_r).\end{aligned}$$

This, the supports of the measures P_{ζ_j} and the minimality of the support prove the theorem. □

8 Proof of universality

Theorems 2 and 3 easily follows from Theorems 6 and 7 as well as the Mergelyan theorem [22] on the approximation of analytic functions by polynomials. For convenience, we recall the latter beautiful theorem.

Lemma 12. *Suppose that $K\subset\mathbb{C}$ is a compact set with connected complements, and $g(s)$ is a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon>0$, there exists a polynomial $p(s)$ such that* $\sup_{s\in K}|g(s)-p(s)|<\varepsilon$.

Proof of Theorem 2.

1. Lemma 12 implies the existence of polynomials $p_1(s),\dots,p_r(s)$ such that

$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\bigl|f_j(s)-\mathrm{e}^{p_j(s)}\bigr|<\frac{\varepsilon}{2}. \tag{22}$$

Consider the set

$$G_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D)\colon \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| g_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2} \right\}.$$

By Theorem 7, the set G_ε is an open neighbourhood of the element $(\mathrm{e}^{p_1(s)}, \dots, \mathrm{e}^{p_r(s)})$ of the support of the measure $P_{\underline{\zeta}}$. Thus, by a property of the support,

$$P_{\underline{\zeta}}(G_\varepsilon) > 0. \tag{23}$$

Therefore, Theorem 6, together with equivalent of weak convergence of probability measures in terms of open sets [4, Thm. 2.1], gives

$$\liminf_{T \to \infty} P_T^1(G_\varepsilon) \geqslant P_{\underline{\zeta}}(G_\varepsilon) > 0.$$

This, the definitions of P_T^1 and G_ε, and (22) prove the first part of the theorem.

2. Define one more set

$$\hat{G}_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D)\colon \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| g_j(s) - f_j(s) \right| < \varepsilon \right\}.$$

The boundary $\partial \hat{G}_\varepsilon$ of $\hat{G}_\varepsilon$ lies in the set

$$\left\{ (g_1, \dots, g_r) \in H^r(D)\colon \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| g_j(s) - f_j(s) \right| = \varepsilon \right\},$$

therefore, $\partial \hat{G}_{\varepsilon_1} \cap \partial \hat{G}_{\varepsilon_2} = \emptyset$ for different positive ε_1 and ε_2. From this we have that $P_{\underline{\zeta}}(\partial \hat{G}_\varepsilon) = 0$, i.e., the set $\hat{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}$ for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 6, together with equivalent of weak convergence of probability measures in terms of continuity sets [4, Thm. 2.1], shows that

$$\lim_{T \to \infty} P_T^1(\hat{G}_\varepsilon) = P_{\underline{\zeta}}(\hat{G}_\varepsilon) \tag{24}$$

for all but at most countably many $\varepsilon > 0$. In view of (22), the inclusion $G_\varepsilon^r \subset \hat{G}_\varepsilon^r$ follows. Thus, by (23), we have $P_{\underline{\zeta}}(\hat{G}_\varepsilon) > 0$. This, the definitions of P_T^1 and $\hat{G}_\varepsilon$, and (24) prove the second assertion of the theorem. □

Proof of Theorem 3. We repeat the proof of Theorem 2 with P_T^r in place of P_T^1. □

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