

ON SOME EXTENSIONS OF THE A-MODEL

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Abstract. The A-model for finite rank singular perturbations of class $\mathfrak{H}_{-m-2} \setminus \mathfrak{H}_{-m-1}$, $m \in \mathbb{N}$, is considered from the perspective of boundary relations. Assuming further that the Hilbert spaces $(\mathfrak{H}_n)_{n \in \mathbb{Z}}$ admit an orthogonal decomposition $\mathfrak{H}_n^- \oplus \mathfrak{H}_n^+$, with the corresponding projections satisfying $P_{n+1}^\pm \subseteq P_n^\pm$, nontrivial extensions in the A-model are constructed for the symmetric restrictions in the subspaces.

Keywords: finite rank higher order singular perturbation, cascade (A) model, peak model, Hilbert space, scale of Hilbert spaces, Pontryagin space, ordinary boundary triple, Krein Q -function, Weyl function, gamma field, symmetric operator, proper extension, resolvent.

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1. INTRODUCTION

Consider a lower semibounded self-adjoint operator L in a Hilbert space \mathfrak{H}_0 . Let $\mathfrak{H}_{n+1} \subseteq \mathfrak{H}_n$, $n \in \mathbb{Z}$, be the scale of Hilbert spaces associated with L . Let also $\{\varphi_\sigma\}$ be the family of linearly independent functionals of class $\mathfrak{H}_{-m-2} \setminus \mathfrak{H}_{-m-1}$, $m \in \mathbb{N}$, where σ ranges over an index set \mathcal{S} of dimension $d \in \mathbb{N}$. Then, the symmetric restriction $L_{\min} \subseteq L$ to the domain of $f \in \mathfrak{H}_{m+2}$ such that $\langle \varphi_\sigma, f \rangle = 0$, for all σ , is an essentially self-adjoint operator in \mathfrak{H}_0 . Sequentially, traditional methods, see e.g. [2, 20], for describing nontrivial extensions of L_{\min} (i.e. perturbations of L) in \mathfrak{H}_0 are insufficient. The classical examples of higher order singular perturbations are the point-interactions modeled by the Dirac distribution and its derivatives.

To construct nontrivial realizations of L_{\min} in Hilbert or Pontryagin spaces, one considers instead the so-called cascade (A or B) models [15–17, 25, 27] and the peak model [24, 26]. In these models the Weyl (or Krein Q -) function is the sum of a Nevanlinna function associated with L_{\min} in \mathfrak{H}_m and a generalized Nevanlinna function associated with a certain multiplication operator in a reproducing kernel Pontryagin space [5, Theorem 4.10]; more on reproducing kernel spaces can be found in [3, 6, 7, 10]. Successively, singular perturbations are interpreted by means of the

compression to the reference space \mathfrak{H}_0 of the resolvent of an appropriate extension in the model space.

Here we study the cascade A-model for rank- d higher order singular perturbations. More precisely, for a specific choice of model parameters, we extend the main results obtained in [15] to the case of an arbitrary $d \in \mathbb{N}$ (see Theorem 3.2). The exposition utilizes the techniques based on the notion of boundary triples [11–14]. Then, by assuming that the Hilbert space \mathfrak{H}_n is expressed as the Hilbert sum $\mathfrak{H}_n^- \oplus \mathfrak{H}_n^+$ of its subspaces \mathfrak{H}_n^\pm , we examine nontrivial realizations that account for the above described Hilbert space decomposition (Theorem 7.3). We assume that the corresponding orthogonal projections P_n^\pm from \mathfrak{H}_n onto \mathfrak{H}_n^\pm satisfy the inclusions $P_{n+1}^\pm \subseteq P_n^\pm$. This further implies that the subspaces \mathfrak{H}_n^\pm reduce the self-adjoint restriction to \mathfrak{H}_{n+2} of L (Theorem 5.6). As a natural consequence of our hypothesis is that the Weyl function associated with the symmetric operator L_{\min} in \mathfrak{H}_m is the sum of the Weyl functions associated with the symmetric restrictions to \mathfrak{H}_m^\pm of L_{\min} .

The projection of the model to the subspaces just described has a natural application in quantum mechanics when, for example, one wishes to account for the contribution to the eigenvalues of antisymmetric (resp. symmetric) eigenfunctions. For instance, if one takes L such that $\mathfrak{H}_n = W_2^n \otimes \mathbb{C}^4$, where W_2^n is the Sobolev space (Example 4.3), then the projections P_n^- and P_n^+ onto the spaces of antisymmetric spin states, $W_2^n \otimes \mathbb{C}^1$, and onto the spaces of symmetric spin states, $W_2^n \otimes \mathbb{C}^3$, satisfy our hypothesis. However, a concrete application of the present model will be demonstrated elsewhere.

Another motivation for considering the A-model, as opposed to the peak model, arises from an attempt to elude a too restrictive condition imposed on the Gram matrix $\mathcal{G} = (\mathcal{G}_{\sigma_j, \sigma'_j}) \in [\mathbb{C}^{m \times m}]$ of the peak model; namely, \mathcal{G} must be diagonal in $j \in \{1, \dots, m\}$. Although initially contemplated as an advantageous feature [26], this restriction is not satisfied for some operators L , for $m > 1$, for a simple reason that the eigenvectors of the triplet adjoint of L_{\min} for the Hilbert triple $\mathfrak{H}_m \subseteq \mathfrak{H}_0 \subseteq \mathfrak{H}_{-m}$ are not necessarily orthogonal for distinct eigenvalues (Example 3.4).

2. PRELIMINARIES

Let A be a densely defined, closed, symmetric operator in a Pontryagin space \mathfrak{H} (see e.g. [4, Sec. 1.9]) with an indefinite metric $[\cdot, \cdot]_{\mathfrak{H}}$. Let A^* be the adjoint in \mathfrak{H} of A . A triple $(\mathcal{H}, \Gamma_0, \Gamma_1)$, where $\mathcal{H} = (\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a Hilbert space and $\Gamma: f \mapsto (\Gamma_0 f, \Gamma_1 f)$ is the operator from $\text{dom } A^*$ to $\mathcal{H}^2 (:= \mathcal{H} \times \mathcal{H})$, is called an ordinary boundary triple (OBT) for A^* if Γ is surjective and the Green identity holds:

$$[f, g]_{A^*} := [f, A^* g]_{\mathfrak{H}} - [A^* f, g]_{\mathfrak{H}} = \langle \Gamma_0 f, \Gamma_1 g \rangle_{\mathcal{H}} - \langle \Gamma_1 f, \Gamma_0 g \rangle_{\mathcal{H}}$$

for all $f, g \in \text{dom } A^*$; see e.g. [8, Definition 2.1]. It is shown that an OBT for A^* in a Pontryagin space (or more generally in a Krein space) exists iff A admits a self-adjoint extension in \mathfrak{H} (cf. [5, Proposition 3.4], [9, p. 192]).

If the assumption on the density of $\text{dom } A$ is dropped off, that is, if A^* is a linear relation [18, 22], then an OBT $(\mathcal{H}, \Gamma_0, \Gamma_1)$ for A^* is defined by considering $\Gamma_i, i \in \{0, 1\}$, as a mapping from A^* onto \mathcal{H} . Sequentially, the Green identity reads

$$[f, g']_{\mathfrak{H}} - [f', g]_{\mathfrak{H}} = \langle \Gamma_0 \widehat{f}, \Gamma_1 \widehat{g} \rangle_{\mathcal{H}} - \langle \Gamma_1 \widehat{f}, \Gamma_0 \widehat{g} \rangle_{\mathcal{H}}$$

for $\widehat{f} = (f, f'), \widehat{g} = (g, g') \in A^*$. The reader may also consult [9, Definition 6], as well as [21, Definition 2.3], [14, Definition 7.11] in the Hilbert space case. In what follows we frequently identify operators with their graphs. Then the present definition of an OBT reduces to the previous definition as long as A becomes densely defined.

A proper extension A_{Θ} of A , i.e. such that $A \subseteq A_{\Theta} \subseteq A^*$, is uniquely determined by a linear relation Θ in \mathcal{H} via $\Theta = \Gamma A_{\Theta}$ with $A_{\Theta} = \{\widehat{f} \in A^* \mid \Gamma \widehat{f} \in \Theta\}$; see e.g. [9, Proposition 2], [21, Proposition 2.5], [14, Proposition 7.12], [8, Proposition 2.1]. In particular, a distinguished self-adjoint extension $A_0 := A^*|_{\ker \Gamma_0}$ corresponds to a self-adjoint linear relation $\Theta = \{0\} \times \mathcal{H}$ (and similarly for the transversal one, corresponding to $\Theta = \mathcal{H} \times \{0\}$). A self-adjoint linear relation in a Krein (or Pontryagin) space may have an empty resolvent set (see e.g. [5, Example 3.7]). However, if there exists at least one self-adjoint extension of A , say \widetilde{A} , whose resolvent set $\text{res } \widetilde{A}$ is nonempty, then there exists an OBT for A^* such that $\widetilde{A} = A_0$.

Let A be a closed symmetric operator as above. Let $\mathfrak{N}_z(A^*) := \ker(A^* - z), z \in \mathbb{C}$, denote the eigenspace of a linear relation A^* (and similarly for other linear relations and operators). Let $\widehat{\mathfrak{N}}_z(A^*)$ be the set of the pairs $(f_z, z f_z)$ with $f_z \in \mathfrak{N}_z(A^*)$. Let also π_1 denote the orthogonal projection in the Hilbert sum of a Hilbert space with itself onto the first factor. Assume that the resolvent set $\text{res } A_0 \neq \emptyset$. The γ -field γ and the Weyl function M corresponding to the OBT $(\mathcal{H}, \Gamma_0, \Gamma_1)$ for A^* are bounded operator valued functions defined by [9, Definition 7], [21, Definition 2.6]

$$\gamma(z) := \pi_1 \widehat{\gamma}(z), \quad \widehat{\gamma}(z) := (\Gamma_0|_{\widehat{\mathfrak{N}}_z(A^*)})^{-1}, \quad M(z) := \Gamma_1 \widehat{\gamma}(z)$$

for $z \in \text{res } A_0$. Then the resolvent of a closed proper extension A_{Θ} , i.e. such that Θ is closed, is represented by the Krein–Naimark resolvent formula (see e.g. [9, Theorem 4], [8, Theorem 2.1])

$$(A_{\Theta} - z)^{-1} = (A_0 - z)^{-1} + \gamma(z)(\Theta - M(z))^{-1}\gamma(\bar{z})^*$$

for $z \in \text{res } A_0 \cap \text{res } A_{\Theta}$. Moreover, $z \in \text{res } A_{\Theta}$ iff $0 \in \text{res}(\Theta - M(z))$.

Let $\mathfrak{H} = (\mathfrak{H}, [\cdot, \cdot]_{\mathfrak{H}})$ be a Krein (or in particular Pontryagin) space, let $\mathcal{H} = (\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be a Hilbert space. Consider a linear relation $\Gamma \subseteq \mathfrak{H}^2 \times \mathcal{H}^2$. Let $\Gamma^{[+]}$ be its Krein space adjoint:

$$\Gamma^{[+]} := \{((h_o, h'_o), (g, g')) \in \mathcal{H}^2 \times \mathfrak{H}^2 \mid \forall ((f, f'), (h, h')) \in \Gamma : [f, g']_{\mathfrak{H}} - [f', g]_{\mathfrak{H}} = \langle h, h'_o \rangle_{\mathcal{H}} - \langle h', h_o \rangle_{\mathcal{H}}\}.$$

Then Γ is said to be an isometric (resp. unitary) linear relation if the inverse linear relation $\Gamma^{-1} \subseteq \Gamma^{[+]}$ (resp. $\Gamma^{-1} = \Gamma^{[+]}$). If Γ is unitary and additionally single-valued (i.e. an operator identified with its graph), then by [12, Corollary 2.4(i)] $\overline{\text{ran}} \Gamma = \mathcal{H}^2$

(the closure of the range). If, moreover, $\text{dom } \Gamma$ is closed, then also $\text{ran } \Gamma$ is closed, and is given by $\text{ran } \Gamma = \mathcal{H}^2$ ([12, Corollary 2.4(iii)]).

Throughout we use quite standard notation for the domain $\text{dom } A$, the range $\text{ran } A$, the kernel $\ker A$, and the multivalued part $\text{mul } A$ of a linear relation A . The resolvent set of A is denoted by $\text{res } A$, the point spectrum by $\sigma_p(A)$.

3. THE A-MODEL FOR FINITE RANK PERTURBATIONS

Let $\mathfrak{H}_{n+1} \subseteq \mathfrak{H}_n$, $n \in \mathbb{Z}$, be the scale of Hilbert spaces associated with a lower semibounded self-adjoint operator L defined in the reference Hilbert space \mathfrak{H}_0 with domain $\text{dom } L = \mathfrak{H}_2$. The scalar product in \mathfrak{H}_n is defined via the scalar product $\langle \cdot, \cdot \rangle_0$ in \mathfrak{H}_0 by scaling according to

$$\langle \cdot, \cdot \rangle_n := \langle b_n(L)^{1/2} \cdot, b_n(L)^{1/2} \cdot \rangle_0, \quad b_n(L) := (L - z_1)^n.$$

The number $z_1 \in \text{res } L \cap \mathbb{R}$ is fixed and referred to as the model parameter. Let us mention that the above definition of the \mathfrak{H}_n -scalar product allows us to avoid extra technicalities arising when, for example, one chooses $b_n(L)$ as the product of $(L - z_j)$ for $j \in \{1, \dots, n\}$ for not necessarily identical model parameters z_j , as is done in [15] (where $z_j = -a_j$), or when, on top of that, one assumes L not necessarily semibounded, in which case one should put $|L|$ in $b_n(L)$ instead of L . On the other hand, our definition of the scalar product predefines the inner structure of the model space (to be defined later); namely, it is shown in [15, Theorem 3.2(iii)] for $d = 1$ that the present choice of the model parameters (i.e. $a_j = -z_1$ for all j) leads to an indefinite inner product space, as the model space. Let us moreover advertise that the current definition of the unitary operator $b_n(L)^{1/2}$ (from \mathfrak{H}_n to \mathfrak{H}_0) is not allowed in the peak model [26], which is a purely Hilbert space model (cf. [15, Theorem 3.2(ii)]).

To $L = L_0$ one associates an operator $L_n := L|_{\mathfrak{H}_{n+2}}$ in \mathfrak{H}_n . Then L_n is self-adjoint in \mathfrak{H}_n , and moreover $L_{n+1} \subset L_n$ and $\text{res } L_n = \text{res } L$ (cf. Section 5). For notational simplicity we drop-off the subscript when no confusion can arise.

Let us fix $m \in \mathbb{N}$. Let L_{\max} denote the triplet adjoint of L_{\min} for the Hilbert triple $\mathfrak{H}_m \subset \mathfrak{H}_0 \subset \mathfrak{H}_{-m}$; see also [15, Theorem 2.1], [26, Definition 3.1], [24, Proposition 4.2]. The operator L_{\max} extends L_{-m+2} to (the direct sum)

$$\text{dom}(L_{\max}) = \mathfrak{H}_{-m+2} \dot{+} \mathfrak{N}_z(L_{\max}), \quad z \in \text{res } L.$$

$\mathfrak{N}_z(L_{\max})$ is the linear span of the singular elements $\{g_\sigma(z) \in \mathfrak{H}_{-m} \setminus \mathfrak{H}_{-m+1}\}$, each being defined so that $b_m(L)^{-1}g_\sigma(z) \in \mathfrak{H}_m \setminus \mathfrak{H}_{m+1}$ is a deficiency element of the adjoint L_{\min}^* in \mathfrak{H}_m of a densely defined, closed, symmetric operator L_{\min} in \mathfrak{H}_m with defect numbers (d, d) . Let us recall that the domain of L_{\min} is parametrized via the family of linearly independent functionals $\{\varphi_\sigma \in \mathfrak{H}_{-m-2} \setminus \mathfrak{H}_{-m-1}\}$ according to $\langle \varphi_\sigma, f \rangle = 0$ for $f \in \mathfrak{H}_{m+2}$; the duality pairing $\langle \cdot, \cdot \rangle$ between \mathfrak{H}_{-m-2} and \mathfrak{H}_{m+2} is defined via the \mathfrak{H}_0 -scalar product in a usual way (cf. [2, Eq. (1.17)]). In the sequel we also use the vector notation $\langle \varphi, \cdot \rangle = (\langle \varphi_\sigma, \cdot \rangle): \mathfrak{H}_{m+2} \rightarrow \mathbb{C}^d$, and similarly for other duality pairings. In terms of the functionals $\{\varphi_\sigma\}$ the eigenvectors of L_{\max} are then given (in the generalized sense) by $g_\sigma(z) := (L - z)^{-1}\varphi_\sigma$.

As the space \mathfrak{H}_{-m} in which L_{\max} acts is too large, following the lines of [15] one further considers L_{\max} in a finite-dimensional extension of \mathfrak{H}_m , referred to as an intermediate (or model) space. We now discuss the construction of the space in more detail.

Consider an md -dimensional linear space

$$\mathfrak{K}_A := \text{span}\{h_\alpha \mid \alpha = (\sigma, j) \in \mathcal{S} \times J\}, \quad J := \{1, 2, \dots, m\}$$

(\mathcal{S} is an index set of dimension d) spanned by the elements

$$h_{\sigma j} := (L - z_1)^{-j} \varphi_\sigma \in \mathfrak{H}_{-m-2+2j} \setminus \mathfrak{H}_{-m-1+2j}.$$

Note that $h_{\sigma 1} = g_\sigma(z_1) \in \mathfrak{N}_{z_1}(L_{\max})$. An element $k \in \mathfrak{K}_A \subseteq \mathfrak{H}_{-m}$ is thus of the form

$$k = \sum_{\alpha} d_\alpha(k) h_\alpha, \quad d_\alpha(k) \in \mathbb{C}.$$

Since the system $\{h_\alpha\}$ is linearly independent, the Gram matrix

$$\tilde{\mathcal{G}}_A = ([\tilde{\mathcal{G}}_A]_{\alpha\alpha'}) \in [\mathbb{C}^{md}], \quad [\tilde{\mathcal{G}}_A]_{\alpha\alpha'} := \langle h_\alpha, h_{\alpha'} \rangle_{-m}$$

is positive definite, and one establishes a bijective correspondence

$$\mathfrak{K}_A \ni k \leftrightarrow d(k) = (d_\alpha(k)) \in \mathbb{C}^{md}.$$

Observe that $\mathfrak{K}_A \cap \mathfrak{H}_{m-1} = \{0\}$.

Define a linear space

$$\mathcal{H}_A := (\mathfrak{H}_m \dot{+} \mathfrak{K}_A, [\cdot, \cdot]_A)$$

with an indefinite metric

$$[f + k, f' + k']_A := \langle f, f' \rangle_m + \langle d(k), \mathcal{G}_A d(k') \rangle_{\mathbb{C}^{md}}$$

for $f, f' \in \mathfrak{H}_m$; $k, k' \in \mathfrak{K}_A$. An Hermitian matrix $\mathcal{G}_A = ([\mathcal{G}_A]_{\alpha\alpha'}) \in [\mathbb{C}^{md}]$ is referred to as the Gram matrix of the A-model. The model space \mathcal{H}_A is a Hilbert space if $\mathcal{G}_A \geq 0$ and a Pontryagin space otherwise. Let also

$$\mathcal{H}'_A := (\mathfrak{H}_m \oplus \mathbb{C}^{md}, [\cdot, \cdot]'_A)$$

with an indefinite metric

$$[(f, \xi), (f', \xi')]'_A := \langle f, f' \rangle_m + \langle \xi, \mathcal{G}_A \xi' \rangle_{\mathbb{C}^{md}}$$

for $(f, \xi), (f', \xi') \in \mathfrak{H}_m \oplus \mathbb{C}^{md}$. The isometric isomorphism (unitary operator) from \mathcal{H}_A onto \mathcal{H}'_A , realized via the above established bijective correspondence $\mathfrak{K}_A \leftrightarrow \mathbb{C}^{md}$, is denoted by U_A .

The construction of nontrivial extensions to \mathcal{H}_A of L_{\min} relies upon the following lemma; cf. [15, Eq. (2.3)].

Lemma 3.1. *The restriction to \mathcal{H}_A of L_{\max} is the operator A_{\max} given by*

$$\begin{aligned} \text{dom } A_{\max} &= \left\{ f^\# + h_{m+1}(c) + k \mid f^\# \in \mathfrak{H}_{m+2}, k \in \mathfrak{K}_A, h_{m+1}(c) := \sum_{\sigma} c_{\sigma} h_{\sigma, m+1}, \right. \\ &\quad \left. c = (c_{\sigma}) \in \mathbb{C}^d, h_{\sigma, m+1} := b_{m+1}(L)^{-1} \varphi_{\sigma} \in \mathfrak{H}_m \setminus \mathfrak{H}_{m+1} \right\}, \\ A_{\max}(f^\# + h_{m+1}(c) + k) &= Lf^\# + z_1 h_{m+1}(c) + \tilde{k}, \quad \tilde{k} \in \mathfrak{K}_A, \\ d(\tilde{k}) &:= \mathfrak{M}_d d(k) + \eta(c), \quad \eta(c) := (\delta_{jm} c_{\sigma}) \in \mathbb{C}^{md}, \end{aligned}$$

where the matrix $\mathfrak{M}_d := \mathfrak{M} \oplus \dots \oplus \mathfrak{M}$ (d times) is the matrix direct sum of d matrices $\mathfrak{M} = (\mathfrak{M}_{jj'}) \in [\mathbb{C}^m]$ defined by

$$\mathfrak{M}_{jj'} := \delta_{jj'} z_1 + \delta_{j+1, j'}, \quad j \in J \setminus \{m\}, j' \in J$$

and $\mathfrak{M}_{mj'} := \delta_{j'm} z_1, j' \in J$. For $m = 1$, one puts $\mathfrak{M} := z_1$.

Proof. By definition, the action of L_{\max} on $f + k \in \mathfrak{H}_m \dot{+} \mathfrak{K}_A$ is given (in the generalized sense) by

$$\begin{aligned} L_{\max}(f + k) &= Lf + \sum_{\sigma} z_1 d_{\sigma 1}(k) h_{\sigma 1} + \sum_{\sigma} \sum_{j=2}^m d_{\sigma j}(k) L(L - z_1)^{-j} \varphi_{\sigma} \\ &= Lf + z_1 k + \sum_{\sigma} \sum_{j=1}^{m-1} d_{\sigma, j+1}(k) h_{\sigma j}. \end{aligned}$$

Now $Lf \in \mathfrak{H}_{m-2}$, thus the range restriction $L_{\max}(f + k) \in \mathfrak{H}_m \dot{+} \mathfrak{K}_A$ implies that f is of the form $f^\# + g$ for some $f^\# \in \mathfrak{H}_{m+2}$ and $g \in \mathfrak{H}_m$ such that $Lg \in \mathcal{H}_A$. By noting that $Lh_{m+1}(c) = z_1 h_{m+1}(c) + h_m(c)$ ($h_m(c) \in \mathfrak{K}_A$ is defined similar to $h_{m+1}(c)$) for an arbitrary $c \in \mathbb{C}^d$, one concludes that $g = h_{m+1}(c)$, and the required result follows. \square

Now we state the main realization theorem in the A-model.

Theorem 3.2. *Assume that an invertible Hermitian matrix \mathcal{G}_A satisfies the commutation relation*

$$\mathcal{G}_A \mathfrak{M}_d = \mathfrak{M}_d^* \mathcal{G}_A. \tag{3.1}$$

Then the triple $(\mathbb{C}^d, \Gamma_0^A, \Gamma_1^A)$, where $\Gamma^A : f \mapsto (\Gamma_0^A f, \Gamma_1^A f)$ from $\text{dom } A_{\max}$ to $\mathbb{C}^d \times \mathbb{C}^d$ is defined by

$$\begin{aligned} \Gamma_0^A(f^\# + h_{m+1}(c) + k) &:= c, \\ \Gamma_1^A(f^\# + h_{m+1}(c) + k) &:= \langle \varphi, f^\# \rangle - [\mathcal{G}_A d(k)]_m \end{aligned}$$

with

$$[\mathcal{G}_A d(k)]_m := ([\mathcal{G}_A d(k)]_{\sigma m}) \in \mathbb{C}^d$$

and $f^\# \in \mathfrak{H}_{m+2}, k \in \mathfrak{K}_A, c \in \mathbb{C}^d$, is an OBT for the adjoint $A_{\min}^* = A_{\max}$ of a densely defined, closed, symmetric operator $A_{\min} = A_{\max} \upharpoonright_{\ker \Gamma^A}$ in \mathcal{H}_A .

Moreover, for a (closed) linear relation Θ in \mathbb{C}^d , a proper extension A_Θ of A_{\min} is the restriction of A_{\max} to the set of $f \in \text{dom } A_{\max}$ such that $\Gamma^A f \in \Theta$. The Krein–Naimark resolvent formula reads

$$(A_\Theta - z)^{-1} = (A_0 - z)^{-1} + \gamma_A(z)(\Theta - M_A(z))^{-1}\gamma_A(\bar{z})^*$$

for $z \in \text{res } A_0 \cap \text{res } A_\Theta$. The resolvent of a distinguished self-adjoint extension $A_0 := A_{\{0\} \times \mathbb{C}^d}$ is given by

$$(A_0 - z)^{-1} = U_A^*[(L - z)^{-1} \oplus (\mathfrak{M}_d - z)^{-1}]U_A$$

for $z \in \text{res } A_0 = \text{res } L \setminus \{z_1\}$. The γ -field γ_A and the Weyl function M_A corresponding to $(\mathbb{C}^d, \Gamma_0^A, \Gamma_1^A)$ are given by

$$\gamma_A(z)\mathbb{C}^d = \mathfrak{N}_z(A_{\max}) = \left\{ \sum_{\sigma} c_{\sigma} F_{\sigma}(z) \mid c_{\sigma} \in \mathbb{C} \right\}, \quad F_{\sigma}(z) := \frac{g_{\sigma}(z)}{(z - z_1)^m}$$

and

$$M_A(z) = q(z) + r(z) \quad \text{on } \mathbb{C}^d$$

for $z \in \text{res } A_0$. The Krein Q -function q of L_{\min} is defined by

$$q(z) = ([q(z)]_{\sigma\sigma'}) \in [\mathbb{C}^d], \quad [q(z)]_{\sigma\sigma'} := (z - z_1) \langle \varphi_{\sigma}, (L - z)^{-1} h_{\sigma', m+1} \rangle$$

for $z \in \text{res } L$, and the generalized Nevanlinna function r is defined by

$$r(z) = ([r(z)]_{\sigma\sigma'}) \in [\mathbb{C}^d], \quad [r(z)]_{\sigma\sigma'} := - \sum_j \frac{[\mathcal{G}_A]_{\sigma m, \sigma' j}}{(z - z_1)^{m-j+1}}$$

for $z \in \mathbb{C} \setminus \{z_1\}$.

Proof. By Lemma 3.1, the boundary form of A_{\max} is given by

$$[f, g]_{A_{\max}} = \langle d(k), (\mathcal{G}_{\mathfrak{M}} - \mathcal{G}_{\mathfrak{M}}^*)d(k') \rangle_{\mathbb{C}^{md}} + \langle \Gamma_0^A f, \Gamma_1^A g \rangle_{\mathbb{C}^d} - \langle \Gamma_1^A f, \Gamma_0^A g \rangle_{\mathbb{C}^d}$$

with $\mathcal{G}_{\mathfrak{M}} := \mathcal{G}_A \mathfrak{M}_d$, where $f = f^{\#} + h_{m+1}(c) + k \in \text{dom } A_{\max}$, $g = g^{\#} + h_{m+1}(c') + k' \in \text{dom } A_{\max}$, $f^{\#}, g^{\#} \in \mathfrak{H}_{m+2}$, $c, c' \in \mathbb{C}^d$, $k, k' \in \mathfrak{K}_A$. Assuming that

$$\ker \mathcal{G}_A = \{0\} \quad \text{and} \quad \mathfrak{M}_d^* \mathcal{G}_A \mathbb{C}^{md} \subseteq \text{ran } \mathcal{G}_A$$

the adjoint $A_{\min} := A_{\max}^*$ in \mathcal{H}_A is given by

$$\begin{aligned} \text{dom } A_{\min} &= \ker \Gamma^A, \\ A_{\min}(f^{\#} + k) &= Lf^{\#} + \sum_{\alpha} [\mathcal{G}_A^{-1} \mathfrak{M}_d^* \mathcal{G}_A d(k)]_{\alpha} h_{\alpha} \end{aligned}$$

and hence the boundary form of A_{\min} reads

$$[f, g]_{A_{\min}} = \langle d(k), (\mathcal{G}_{\mathfrak{M}}^* - \mathcal{G}_{\mathfrak{M}})d(k') \rangle_{\mathbb{C}^{md}}$$

with $f = f^\# + k \in \text{dom } A_{\min}$ and $g = g^\# + k' \in \text{dom } A_{\min}$ as above. One verifies that the adjoint $A_{\min}^* = A_{\max}$, and hence A_{\max} is closed in \mathcal{H}_A .

If (3.1) holds, the boundary form of A_{\min}^* satisfies an abstract Green identity. Thus, since Γ^A is single-valued and surjective, the triple $(\mathbb{C}^d, \Gamma_0^A, \Gamma_1^A)$ is an OBT for A_{\min}^* .

The eigenvalue equation for A_{\max} yields

$$f^\# = (z - z_1)(L - z)^{-1}h_{m+1}(c), \quad d(k) = -(\mathfrak{M}_d - z)^{-1}\eta(c) \tag{3.2}$$

for $f^\# + h_{m+1}(c) + k \in \text{dom } A_{\max}$ as above. Now

$$[(\mathfrak{M}_d - z)^{-1}\eta(c)]_{\sigma j} = \sum_{\sigma'} [(\mathfrak{M}_d - z)^{-1}]_{\sigma j, \sigma' m} c_{\sigma'}$$

with $c = (c_\sigma) \in \mathbb{C}^d$ and with

$$[(\mathfrak{M}_d - z)^{-1}]_{\sigma j, \sigma' m} = \delta_{\sigma\sigma'} [(\mathfrak{M} - z)^{-1}]_{jm}, \quad [(\mathfrak{M} - z)^{-1}]_{jm} = \frac{-1}{(z - z_1)^{m-j+1}}.$$

Thus, by noting that

$$(L - z)^{-1}(L - z_1)^{-m} + \sum_j (L - z_1)^{-j}(z - z_1)^{-m+j-1} = (L - z)^{-1}(z - z_1)^{-m}$$

one concludes that the eigenvector $f^\# + h_{m+1}(c) + k \in \mathfrak{N}_z(A_{\max})$ is given as stated in the theorem.

Finally, the Weyl function

$$M_A(z)c = \langle \varphi, f^\# \rangle - [\mathcal{G}_A d(k)]_m$$

for $f^\#$ and k as in (3.2). The first term on the right-hand side defines $q(z)c$ and the second term defines $r(z)c$. □

Let us mention that the Q -function q is actually the Weyl function associated with a certain boundary triple for the adjoint L_{\min}^* in \mathfrak{H}_m ; see Corollary 7.4 below. While q is a Nevanlinna function, r is a generalized Nevanlinna function, and the Nevanlinna class [3, 7] depends on the particular choice of the Gram matrix \mathcal{G}_A .

The matrix $\mathcal{G}_{\mathfrak{M}} := \mathcal{G}_A \mathfrak{M}_d$ is Hermitian iff

$$[\mathcal{G}_A]_{\sigma j, \sigma' j'} = 0, \quad [\mathcal{G}_A]_{\sigma j, \sigma' m} = \overline{[\mathcal{G}_A]_{\sigma' m, \sigma j}} = [\mathcal{G}_A]_{\sigma, j+1; \sigma', m-1} \tag{3.3}$$

for $j \in J \setminus \{m\}$, $j' \in \{1, \dots, m - j\}$ and $m \geq 2$. For $m = 1$, however, the matrix $\mathcal{G}_{\mathfrak{M}} = z_1 \mathcal{G}_A$ is automatically Hermitian.

Due to (3.3), several remarks are in order. First one verifies that r is symmetric with respect to the real axis, that is, $r(z)^* = r(\bar{z})$, because $[\mathcal{G}_A]_{\sigma m, \sigma' j} = [\mathcal{G}_A]_{\sigma j, \sigma' m}$ ($j \in J$) by (3.3). Note that $q(z)^* = q(\bar{z})$ is clear from the definition. Next, one observes that the Gram matrix $\tilde{\mathcal{G}}_A$ does not satisfy (3.1) for $m \geq 2$, because $[\tilde{\mathcal{G}}_A]_{\sigma_1, \sigma_1} > 0$. This shows that, in order use Theorem 3.2 for $m \geq 2$, one cannot define the Gram matrix of the A-model in a way that is done in the peak model.

Remark 3.3. Let us recall that in the peak model the parameters $\{a_j\}$ are all necessarily distinct. However, putting $a_j = -z_1 + \delta_{j-1}$ for $\delta_j \neq 0$ and $j \in J \setminus \{1\}$ and $m \geq 2$, and formally taking the limits $\delta_j \rightarrow \delta_{j-1}$, as well as $\delta_1 \rightarrow 0$, one can show by induction that the Q -function associated with the Gram matrix \mathcal{G} of the peak model approaches r , up to $O(\delta_1)$, with $[\mathcal{G}_A]_{\sigma m, \sigma' j} = [\tilde{\mathcal{G}}_A]_{\sigma m, \sigma' j}$. Notice that $[\tilde{\mathcal{G}}_A]_{\sigma m, \sigma' j}$, with $m \geq 2$, satisfies the second relation in (3.3). On the other hand, taking the above described limits, the matrix element $\mathcal{G}_{\sigma 1, \sigma' 2} = [\tilde{\mathcal{G}}_A]_{\sigma 1, \sigma' 1} + O(\delta_1)$, so the requirement that \mathcal{G} must be diagonal in j – which is essential in applying the extension theory of symmetric operators in the peak model – fails for $m \geq 2$. For $m = 1$, both models produce the same Nevanlinna function $r(z) = \mathcal{G}_A / (z_1 - z)$, provided that $\mathcal{G}_A = \tilde{\mathcal{G}}_A (\in [\mathbb{C}^d])$.

Example 3.4. We briefly demonstrate by a concrete example the case when the eigenvectors $\{g_\sigma(z)\}$ of L_{\max} are not orthogonal for distinct z , that is, the example when the peak model cannot be applied. We consider the two-particle Rashba spin-orbit-coupled operator L in $\mathfrak{H}_0 = L^2(\mathbb{R}^6) \otimes \mathbb{C}^4$ with point-interaction between the two cold atoms [23]. The operator is nonseparable in the center-of-mass coordinate system $(x, X) \in \mathbb{R}^3 \times \mathbb{R}^3$ (x is the distance between the two atoms, X is the center-of-mass coordinate) for a nonzero spin-orbit-coupling strength ε . The interaction is modeled by the Dirac distribution $\varphi_\sigma \in \mathfrak{H}_{-4} \setminus \mathfrak{H}_{-3}$ concentrated at $x = 0$: $\langle \varphi_\sigma, f \rangle = N_\sigma f_\sigma(0, X)$, $f = \sum_\sigma f_\sigma \otimes |\sigma\rangle \in \mathfrak{H}_4$, $N_\sigma > 0$ is the normalization constant, $\{|\sigma\rangle\}$ is an orthonormal basis of \mathbb{C}^4 . Thus we have $m = 2$ and $d = 4$. For simplicity, we assume that ε is negligibly small. In this regime L approximates, up to $O(\varepsilon)$, the operator $(-2\Delta_x - \frac{1}{2}\Delta_X) \otimes I_{\mathbb{C}^4}$ (cf. [1, Eq. (8)]), where Δ_x (resp. Δ_X) is the Laplacian in $x \in \mathbb{R}^3$ (resp. $X \in \mathbb{R}^3$). Then the distribution $g_\sigma(z) \in \mathfrak{H}_{-2} \setminus \mathfrak{H}_{-1}$ admits a relatively simple form

$$g_\sigma(z) = -\frac{N_\sigma}{(2\pi)^3} \frac{zK_2(|\cdot - W_0|\sqrt{-z})}{|\cdot - W_0|^2} \otimes |\sigma\rangle, \quad W_0 = (0, X), \quad z \in \mathbb{C} \setminus [0, \infty)$$

where K_2 is the Macdonald function of second order. Because $m = 2$, it suffices to have in the (peak) model two distinct model parameters $z_1, z_2 < 0$ (or else $a_1, a_2 > 0$). Because now $b_2(L) = (L - z_1)(L - z_2)$, the Gram matrix element $\mathcal{G}_{\sigma 1, \sigma 2}$ reads

$$\begin{aligned} \mathcal{G}_{\sigma 1, \sigma 2} &:= \langle g_\sigma(z_1), g_\sigma(z_2) \rangle_{-2} = \langle g_\sigma(z_1), b_2(L)^{-1} g_\sigma(z_2) \rangle_0 \\ &= \left\langle \varphi_\sigma, [(L - z_1)(L - z_2)]^{-2} \varphi_\sigma \right\rangle = \left\langle \varphi_\sigma, \frac{\partial^2}{\partial u \partial v} [(L - u)(L - v)]^{-1} \varphi_\sigma \Big|_{\substack{u=z_1 \\ v=z_2}} \right\rangle \\ &= \left\langle \varphi_\sigma, \frac{\partial^2}{\partial u \partial v} \frac{g_\sigma(u) - g_\sigma(v)}{u - v} \Big|_{\substack{u=z_1 \\ v=z_2}} \right\rangle \\ &= -\frac{N_\sigma^2}{(2\pi)^3} \lim_{r \rightarrow 0} \frac{1}{r^2} \frac{\partial^2}{\partial u \partial v} \frac{uK_2(r\sqrt{-u}) - vK_2(r\sqrt{-v})}{u - v} \Big|_{\substack{u=-a_1 \\ v=-a_2}} \\ &= \frac{N_\sigma^2}{(2\pi)^3 2^4} \frac{2a_1 a_2 \log(a_1/a_2) - a_1^2 + a_2^2}{(a_2 - a_1)^3} \end{aligned}$$

up to $O(\varepsilon^2)$ (a more accurate computation of $\mathcal{G}_{\sigma 1, \sigma 2}$ shows that the term $O(\varepsilon)$ vanishes).

4. PROJECTIONS

In the remaining part of the present paper we develop the A-model in the subspaces

$$\mathcal{H}'_A{}^- := (\mathfrak{H}_m^- \oplus \mathbb{C}^{md}, [\cdot, \cdot]'_A), \quad \mathcal{H}'_A{}^+ := (\mathfrak{H}_m^+ \oplus \mathbb{C}^{md}, [\cdot, \cdot]'_A)$$

of \mathcal{H}'_A , by assuming that the Hilbert space $\mathfrak{H}_m = \mathfrak{H}_m^- \oplus \mathfrak{H}_m^+$ is the Hilbert (orthogonal) sum of its subspaces \mathfrak{H}_m^\pm . The analogue of Theorem 3.2, in the case when $\mathfrak{H}_{n+1}^\pm \subseteq \mathfrak{H}_n^\pm$ ($\forall n \in \mathbb{Z}$) densely, is stated in Theorem 7.3. First we discuss the properties of the projections that we use later on, then we consider the restrictions to \mathfrak{H}_n^\pm of L_n , and then finally we describe the min-max operators defined in $\mathcal{H}'_A{}^\pm$. The principal difference between the case of the minimal operator A_{\min} considered in \mathcal{H}_A and its analogue A_{\min}^- (resp. A_{\min}^+) considered in $\mathcal{H}'_A{}^-$ (resp. $\mathcal{H}'_A{}^+$) is that A_{\min}^- (resp. A_{\min}^+) becomes nondensely defined in general, that is, the corresponding maximal operator A_{\max}^- (resp. A_{\max}^+) is a linear relation.

Let P_n^- be an orthogonal projection in \mathfrak{H}_n onto a subspace $\mathfrak{H}_n^- \subseteq \mathfrak{H}_n$ and let $P_n^+ := I_{\mathfrak{H}_n} - P_n^-$, an orthogonal projection in \mathfrak{H}_n onto $\mathfrak{H}_n^+ := (\mathfrak{H}_n^-)^\perp$. Here and elsewhere the subscript in $\perp_{\mathfrak{H}_n}$ indicates with respect to which Hilbert space one takes the orthogonal complement.

Lemma 4.1. P_n^- is an orthogonal projection in \mathfrak{H}_n onto a subspace \mathfrak{H}_n^- iff

$$P_0^-(n) := b_n(L)^{1/2} P_n^- b_n(L)^{-1/2}$$

is an orthogonal projection in \mathfrak{H}_0 onto a subspace

$$\mathfrak{H}_0^-(n) := P_0^-(n) \mathfrak{H}_0 = b_n(L)^{1/2} \mathfrak{H}_n^-.$$

If this is the case, then

$$P_0^+(n) := I_{\mathfrak{H}_0} - P_0^-(n) = b_n(L)^{1/2} P_n^+ b_n(L)^{-1/2}$$

is an orthogonal projection in \mathfrak{H}_0 onto a subspace

$$\mathfrak{H}_0^+(n) := \mathfrak{H}_0^-(n)^{\perp_{\mathfrak{H}_0}} = P_0^+(n) \mathfrak{H}_0 = b_n(L)^{1/2} \mathfrak{H}_n^+.$$

Proof. Because

$$P_0^-(n)^2 = b_n(L)^{1/2} (P_n^-)^2 b_n(L)^{-1/2}$$

$P_0^-(n)$ is a projection iff so is P_n^- .

We show that the adjoint $P_0^-(n)^*$ of $P_0^-(n)$ in \mathfrak{H}_0 is given by

$$P_0^-(n)^* = b_n(L)^{1/2} P_n^{-*} b_n(L)^{-1/2} \tag{4.1}$$

on \mathfrak{H}_0 , where P_n^{-*} is the adjoint of P_n^- in \mathfrak{H}_n ; then it follows that $P_0^-(n)$ is self-adjoint in \mathfrak{H}_0 iff so is P_n^- in \mathfrak{H}_n : The graph of the adjoint $P_0^-(n)^*$ in \mathfrak{H}_0 consists of $(y, x) \in \mathfrak{H}_0^2$ such that $(\forall u \in \mathfrak{H}_0)$

$$\langle u, x \rangle_0 = \langle P_0^-(n)u, y \rangle_0.$$

Every u is of the form $u = b_n(L)^{1/2}f$ with some $f \in \mathfrak{H}_n$. Then

$$\langle u, x \rangle_0 = \langle f, b_n(L)^{-1/2}x \rangle_n$$

and

$$\langle P_0^-(n)u, y \rangle_0 = \langle P_n^- f, b_n(L)^{-1/2}y \rangle_n = \langle f, P_n^- * b_n(L)^{-1/2}y \rangle_n$$

from which the claim follows. The remaining statements are verified straightforwardly. \square

The present lemma allows us to freely transfer between the \mathfrak{H}_n -space representation and the \mathfrak{H}_0 -space representation. In particular $\mathfrak{H}_0^-(0) = \mathfrak{H}_0^-$, but in general $\mathfrak{H}_0^-(n) \neq \mathfrak{H}_0^-$ for $n \neq 0$. The equality holds for all n iff

$$P_n^- = b_n(L)^{-1/2}P_0^-b_n(L)^{1/2} \tag{4.2}$$

on \mathfrak{H}_n ; in this case one would have $\mathfrak{H}_{n+l}^- = b_l(L)^{-1/2}\mathfrak{H}_n^-$ for $l \in \mathbb{N}_0$ (cf. Example 4.5). Moreover, $P_0^\pm(n)P_0^\mp(n+l) \neq 0$ in general. However, the product of projections vanishes for $l \in 2\mathbb{Z}$, provided that $P_{n+1}^- \subseteq P_n^-$; see Lemma 4.4 below.

Let $n \in \mathbb{Z}$, $l \in \mathbb{N}_0$ as above and let

$$\mathfrak{H}_{n,l}^- := P_n^- \mathfrak{H}_{n+l}.$$

Throughout we assume that

$$\mathfrak{H}_{n,l}^- \subseteq \mathfrak{H}_{n+l}.$$

Then

$$\mathfrak{H}_{n,l}^- = \mathfrak{H}_n^- \cap \mathfrak{H}_{n+l}.$$

The latter equality follows from the following observations. The set $\mathfrak{H}_n^- \cap \mathfrak{H}_{n+l}$ consists of $f \in \mathfrak{H}_n^-$ such that $f \in \mathfrak{H}_{n+l}$. Then $P_n^- f = f \in \mathfrak{H}_{n,l}^-$, and therefore $\mathfrak{H}_n^- \cap \mathfrak{H}_{n+l}$ is the set of $f \in \mathfrak{H}_{n,l}^-$ such that $f \in \mathfrak{H}_{n+l}$. By the above assumption this yields $\mathfrak{H}_n^- \cap \mathfrak{H}_{n+l} = \mathfrak{H}_{n,l}^-$.

Using the definition of the projection $P_0^-(n)$ it follows that

$$\mathfrak{H}_{n,l}^- = b_n(L)^{-1/2}\mathfrak{H}_l^-(n), \quad \mathfrak{H}_l^-(n) := P_0^-(n)\mathfrak{H}_l = \mathfrak{H}_0^-(n) \cap \mathfrak{H}_l$$

and hence $\mathfrak{H}_l^-(n)$ is a subset of \mathfrak{H}_l . Similarly, one defines $\mathfrak{H}_{n,l}^+ := P_n^+ \mathfrak{H}_{n+l}$ and $\mathfrak{H}_l^+(n) := P_0^+(n)\mathfrak{H}_l$, with the assumption $\mathfrak{H}_{n,l}^+ \subseteq \mathfrak{H}_{n+l}$. We note that

$$P_0^s(n)P_0^{s'}(n') = P_0^{s'}(n')P_0^s(n), \quad s, s' \in \{-, +\}, \quad n, n' \in \mathbb{Z}$$

and that

$$\mathfrak{H}_l^s(n) \cap \mathfrak{H}_l^{s'}(n') = \mathfrak{H}_l^s(n) \cap \mathfrak{H}_l^{s'}(n')$$

for $l \in \mathbb{N}_0$.

In general $\mathfrak{H}_{n,l}^- \neq \mathfrak{H}_{n+l}^-$, but the following holds.

Lemma 4.2. *Let $n \in \mathbb{Z}$, $l \in \mathbb{N}_0$. $\mathfrak{H}_{n,l}^-$ (resp. $\mathfrak{H}_l^-(n)$) is dense in \mathfrak{H}_n^- (resp. $\mathfrak{H}_0^-(n)$). Moreover $\mathfrak{H}_{n,l}^- = \mathfrak{H}_{n+l}^-$ iff \mathfrak{H}_{n+l}^- is dense in \mathfrak{H}_n^- , or equivalently, iff $P_{n+l}^- \subseteq P_n^-$ (in fact, if $\mathfrak{H}_{n+l}^- \subseteq \mathfrak{H}_n^-$ densely, then $\mathfrak{H}_{n+l}^+ \subseteq \mathfrak{H}_n^+$ densely and $P_{n+l}^\pm \subseteq P_n^\pm$; conversely, if $P_{n+l}^- \subseteq P_n^-$, then $P_{n+l}^+ \subseteq P_n^+$ and $\mathfrak{H}_{n+l}^\pm \subseteq \mathfrak{H}_n^\pm$ densely); if this is the case then*

$$P_0^-(n+l) \subseteq b_l(L)^{1/2} P_0^-(n) b_l(L)^{-1/2}$$

and hence $\mathfrak{H}_0^-(n+l) = b_l(L)^{1/2} \mathfrak{H}_l^-(n)$ (and similarly for $P_0^+(n+l)$ and $\mathfrak{H}_0^+(n+l)$).

Proof. The orthogonal complement $(\mathfrak{H}_{n,l}^-)^{\perp_{\mathfrak{H}_n}}$ in \mathfrak{H}_n of $\mathfrak{H}_{n,l}^-$ consists of all $g \in \mathfrak{H}_n$ such that $(\forall f \in \mathfrak{H}_{n+l})$

$$0 = \langle P_n^- f, g \rangle_n = \langle f, P_n^- g \rangle_n.$$

Because \mathfrak{H}_{n+l} is dense in \mathfrak{H}_n , this implies $P_n^- g = 0$; hence $(\mathfrak{H}_{n,l}^-)^{\perp_{\mathfrak{H}_n}} = \mathfrak{H}_n^+$. This shows that $\mathfrak{H}_{n,l}^- \subseteq \mathfrak{H}_n^-$ densely in $\|\cdot\|_n$ -norm. Similarly, the orthogonal complement $\mathfrak{H}_l^-(n)^{\perp_{\mathfrak{H}_0}}$ in \mathfrak{H}_0 of $\mathfrak{H}_l^-(n)$ consists of all $v \in \mathfrak{H}_0$ such that $(\forall u^- \in \mathfrak{H}_l^-(n)) 0 = \langle u^-, v \rangle_0$. Now u^- is of the form $u^- = b_n(L)^{1/2} P_n^- f$ with some $f \in \mathfrak{H}_{n+l}$, so

$$\langle u, v \rangle_0 = \langle b_n(L)^{1/2} P_n^- f, v \rangle_0 = \langle P_n^- f, b_n(L)^{-1/2} v \rangle_n = \langle f, P_n^- b_n(L)^{-1/2} v \rangle_n.$$

This implies $P_n^- b_n(L)^{-1/2} v = 0$, and hence

$$\mathfrak{H}_l^-(n)^{\perp_{\mathfrak{H}_0}} = b_n(L)^{1/2} \mathfrak{H}_n^+ = \mathfrak{H}_0^+(n).$$

One concludes that $\mathfrak{H}_l^-(n) \subseteq \mathfrak{H}_0^-(n)$ densely in $\|\cdot\|_0$ -norm.

Next one shows that $\mathfrak{H}_{n+l}^- \subseteq \mathfrak{H}_n^-$ densely iff $P_{n+l}^- \subseteq P_n^-$. The orthogonal complement $(\mathfrak{H}_{n+l}^-)^{\perp_{\mathfrak{H}_n}}$ in \mathfrak{H}_n of \mathfrak{H}_{n+l}^- is the set of all $g \in \mathfrak{H}_n$ such that $(\forall f \in \mathfrak{H}_{n+l}) 0 = \langle P_{n+l}^- f, g \rangle_n$. If $P_{n+l}^- \subseteq P_n^-$, then one arrives at the previously considered case, namely, $(\mathfrak{H}_{n+l}^-)^{\perp_{\mathfrak{H}_n}} = (\mathfrak{H}_{n,l}^-)^{\perp_{\mathfrak{H}_n}}$; hence $\mathfrak{H}_{n+l}^- \subseteq \mathfrak{H}_n^-$ densely. Moreover, $P_{n+l}^- \subseteq P_n^-$ implies that also $\mathfrak{H}_{n+l}^+ \subseteq \mathfrak{H}_n^+$ densely: $(\mathfrak{H}_{n+l}^+)^{\perp_{\mathfrak{H}_n}}$ is the set of all $g \in \mathfrak{H}_n$ such that $(\forall f \in \mathfrak{H}_{n+l})$

$$0 = \langle P_{n+l}^+ f, g \rangle_n = \langle f, g \rangle_n - \langle P_{n+l}^- f, g \rangle_n,$$

but

$$\langle P_{n+l}^- f, g \rangle_n = \langle P_n^- f, g \rangle_n = \langle f, P_n^- g \rangle_n,$$

so

$$0 = \langle P_{n+l}^+ f, g \rangle_n = \langle f, P_n^+ g \rangle_n.$$

This shows $(\mathfrak{H}_{n+l}^+)^{\perp_{\mathfrak{H}_n}} = \mathfrak{H}_n^-$. Conversely, $(\mathfrak{H}_{n+l}^-)^{\perp_{\mathfrak{H}_n}} = \mathfrak{H}_n^+$ implies that $(\forall f \in \mathfrak{H}_{n+l}) (\forall g \in \mathfrak{H}_n)$

$$0 = \langle P_{n+l}^- f, P_n^+ g \rangle_n = \langle P_n^+ P_{n+l}^- f, g \rangle_n$$

hence $P_n^+ P_{n+l}^- = 0$. On the other hand, $(\mathfrak{H}_{n+l}^-)^{\perp_{\mathfrak{H}_n}} = \mathfrak{H}_n^+$ also implies that $(\mathfrak{H}_{n+l}^+)^{\perp_{\mathfrak{H}_n}} = \mathfrak{H}_n^-$: $(\mathfrak{H}_{n+l}^+)^{\perp_{\mathfrak{H}_n}}$ is the set of all $g \in \mathfrak{H}_n$ such that $(\forall f \in \mathfrak{H}_{n+l})$

$$0 = \langle P_{n+l}^+ f, g \rangle_n = \langle f, g \rangle_n - \langle P_{n+l}^- f, g \rangle_n.$$

Now

$$\langle P_{n+l}^- f, g \rangle_n = \langle P_{n+l}^- f, P_n^- g \rangle_n + \langle P_{n+l}^- f, P_n^+ g \rangle_n$$

and

$$\langle P_{n+l}^- f, P_n^+ g \rangle_n = \langle P_n^+ P_{n+l}^- f, g \rangle_n = 0,$$

so

$$0 = \langle P_{n+l}^+ f, g \rangle_n = \langle f, g \rangle_n - \langle P_{n+l}^- f, P_n^- g \rangle_n = \langle P_{n+l}^- f, g \rangle_n - \langle P_{n+l}^- f, P_n^- g \rangle_n.$$

As a result $(\mathfrak{H}_{n+l}^+)^{\perp \mathfrak{H}_n}$ is the set of all $g \in \mathfrak{H}_n$ such that $(\forall f^- \in \mathfrak{H}_{n+l}^-) 0 = \langle f^-, P_n^+ g \rangle_n$. Because by hypothesis \mathfrak{H}_{n+l}^- is dense in \mathfrak{H}_n^- , this shows $(\mathfrak{H}_{n+l}^+)^{\perp \mathfrak{H}_n} = \mathfrak{H}_n^-$, as claimed. Sequentially, $(\forall f \in \mathfrak{H}_{n+l} \forall g \in \mathfrak{H}_n)$

$$0 = \langle P_{n+l}^+ f, P_n^- g \rangle_n = \langle P_n^- P_{n+l}^+ f, g \rangle_n$$

and hence $P_n^- P_{n+l}^+ = 0$. This together with $P_n^+ P_{n+l}^- = 0$ implies that $P_{n+l}^\pm \subseteq P_n^\pm$.

If $P_{n+l}^- \subseteq P_n^-$ then $\mathfrak{H}_{n,l}^- = \mathfrak{H}_{n+l}^-$ by definition. Assuming the converse, again by definition one gets that $P_n^- \mathfrak{H}_{n+l} = P_{n+l}^- \mathfrak{H}_{n+l}$, i.e. $P_n^- |_{\mathfrak{H}_{n+l}} = P_{n+l}^-$. This shows that $\mathfrak{H}_{n,l}^- = \mathfrak{H}_{n+l}^-$ iff \mathfrak{H}_{n+l}^- is dense in \mathfrak{H}_n^- , or equivalently, iff $P_{n+l}^- \subseteq P_n^-$.

Using $P_{n+l}^- \subseteq P_n^-$, for $u \in \mathfrak{H}_0$

$$\begin{aligned} P_0^-(n+l)u &= b_{n+l}(L)^{1/2} P_{n+l}^- b_{n+l}(L)^{-1/2} u \\ &= b_{n+l}(L)^{1/2} P_n^- b_{n+l}(L)^{-1/2} u \\ &= b_l(L)^{1/2} P_0^-(n) b_l(L)^{-1/2} u \end{aligned}$$

and this completes the proof of the lemma. \square

Example 4.3. Let $H^n = W_2^n(\mathbb{R}^\nu)$, $\nu \in \mathbb{N}$, be the Sobolev space. Then we have $L^2 = L^2(\mathbb{R}^\nu) = H^0$. Let L be such that

$$\mathfrak{H}_n = b_n(L)^{-1/2} (L^2 \otimes \mathbb{C}^4) = H^n \otimes \mathbb{C}^4, \quad n \in \mathbb{Z}$$

and

$$P_n^-(H^n \otimes \mathbb{C}^4) = H^n \otimes \mathbb{C}^1 = \mathfrak{H}_n^-, \quad P_n^+(H^n \otimes \mathbb{C}^4) = H^n \otimes \mathbb{C}^3 = \mathfrak{H}_n^+.$$

Then $P_{n+1}^- \subseteq P_n^-$, and similarly for P_n^+ . The subspaces

$$\mathfrak{H}_0^-(n) = b_n(L)^{1/2} (H^n \otimes \mathbb{C}^1), \quad \mathfrak{H}_0^+(n) = b_n(L)^{1/2} (H^n \otimes \mathbb{C}^3).$$

For $l \in \mathbb{N}_0$, the subset

$$\mathfrak{H}_{n,l}^- = (H^n \otimes \mathbb{C}^1) \cap (H^{n+l} \otimes \mathbb{C}^4) = H^{n+l} \otimes \mathbb{C}^1 = \mathfrak{H}_{n+l}^-$$

is dense in \mathfrak{H}_n^- ; and similarly for $\mathfrak{H}_{n,l}^+ = \mathfrak{H}_{n+l}^+ \subseteq \mathfrak{H}_n^+$. Likewise, the subset

$$\begin{aligned} \mathfrak{H}_l^-(n) &= [b_n(L)^{1/2} (H^n \otimes \mathbb{C}^1)] \cap (H^l \otimes \mathbb{C}^4) \\ &= b_n(L)^{1/2} [(H^n \otimes \mathbb{C}^1) \cap (H^{n+l} \otimes \mathbb{C}^4)] \\ &= b_n(L)^{1/2} (H^{n+l} \otimes \mathbb{C}^1) = b_l(L)^{-1/2} \mathfrak{H}_0^-(n+l) \end{aligned}$$

is dense in $\mathfrak{H}_0^-(n)$, and similarly for $\mathfrak{H}_l^+(n) \subseteq \mathfrak{H}_0^+(n)$.

Due to the dense inclusion $\mathfrak{H}_{n+1} \subseteq \mathfrak{H}_n$ one also has the following result.

Lemma 4.4. *Assume that $P_{n+1}^- \subseteq P_n^-$ for all $n \in \mathbb{Z}$. Then*

$$\mathfrak{H}_0^- = \mathfrak{H}_0^-(2n), \quad \mathfrak{H}_0^-(1) = \mathfrak{H}_0^-(2n + 1).$$

Proof. We show that $\mathfrak{H}_0^+(n) = \mathfrak{H}_0^+(n - 2l)$ for $n \in \mathbb{Z}$, $l \in \mathbb{N}_0$; by relabeling $n - 2l$ by n , the result extends to all $l \in \mathbb{Z}$. Taking the orthogonal complements one deduces an analogous result for $\mathfrak{H}_0^-(n)$.

We use two facts: that $\mathfrak{H}_0^+(n) = \ker P_0^-(n)$ and that $\mathfrak{H}_l \subseteq \mathfrak{H}_0$ densely for $l \in \mathbb{N}_0$. The kernel of $P_0^-(n)$ consists of $u \in \mathfrak{H}_0$ such that $P_0^-(n)u = 0$; this is equivalent to saying that $(\forall v \in \mathfrak{H}_0) \langle v, P_0^-(n)u \rangle_0 = 0$. By Lemma 4.2,

$$P_0^-(n)u = b_l(L)^{1/2}P_0^-(n - l)b_l(L)^{-1/2}u \Rightarrow P_0^-(n - l)b_l(L)^{-1/2}u = 0.$$

Thus for all $v \in \mathfrak{H}_0$

$$\begin{aligned} 0 &= \langle v, P_0^-(n)u \rangle_0 = \langle v, P_0^-(n - l)b_l(L)^{-1/2}u \rangle_0 \\ &= \langle b_l(L)^{-1/2}P_0^-(n - l)v, u \rangle_0 \\ &= \langle P_0^-(n - 2l)b_l(L)^{-1/2}v, u \rangle_0 \quad (\text{by Lemma 4.2}). \end{aligned}$$

Since every v is of the form $v = b_l(L)^{1/2}w$ with some $w \in \mathfrak{H}_l$, it follows that $(\forall w \in \mathfrak{H}_l)$

$$0 = \langle P_0^-(n - 2l)w, u \rangle_0 = \langle w, P_0^-(n - 2l)u \rangle_0.$$

Since $\mathfrak{H}_l \subseteq \mathfrak{H}_0$ densely, the latter implies that $P_0^-(n - 2l)u = 0$; hence

$$\mathfrak{H}_0^+(n) = \ker P_0^-(n) = \ker P_0^-(n - 2l) = \mathfrak{H}_0^+(n - 2l)$$

as claimed. □

Thus, if the hypothesis of Lemma 4.4 holds, then the projections $P_0^-(n)$, $n \in \mathbb{Z}$, are in fact characterized by only two projections: $P_0^- = P_0^-(2n)$ and $P_0^-(1) = P_0^-(2n + 1)$; in this case P_n^- is as in (4.2) for $n \in 2\mathbb{Z}$, and

$$P_n^- = b_n(L)^{-1/2}P_0^-(1)b_n(L)^{1/2}$$

for $n \in 2\mathbb{Z} + 1$. But the converse is not necessarily true in general.

Example 4.5. Let P_n^- be as in (4.2). Then $P_0^-(n) = P_0^-$ for all $n \in \mathbb{Z}$. Let $l \in \mathbb{N}_0$; then

$$\mathfrak{H}_{n,l}^- := P_n^- \mathfrak{H}_{n+l} = b_n(L)^{-1/2}P_0^- b_n(L)^{1/2} \mathfrak{H}_{n+l} = b_n(L)^{-1/2} \mathfrak{H}_{0,l}^-$$

while

$$\mathfrak{H}_{n+l}^- := P_{n+l}^- \mathfrak{H}_{n+l} = b_n(L)^{-1/2} \mathfrak{H}_l^-.$$

Thus $\mathfrak{H}_{n,l}^- = \mathfrak{H}_{n+l}^-$ iff

$$\mathfrak{H}_{0,l}^- (= P_0^- \mathfrak{H}_l) = \mathfrak{H}_l^- (= P_l^- \mathfrak{H}_l)$$

or what is the same, iff $P_0^- \supseteq P_l^-$.

5. PROJECTED OPERATORS

Let $n \in \mathbb{Z}$. By scaling every self-adjoint operator L_n in \mathfrak{H}_n admits the form

$$L_n = b_n(L)^{-1/2} L b_n(L)^{1/2}, \quad L = L_0 \tag{5.1}$$

on $\text{dom } L_n = \mathfrak{H}_{n+2}$. To every L_n one associates densely defined (Lemma 4.2) projected operators

$$L_n^- := P_n^- L_n |_{\mathfrak{H}_{n,2}^-}, \quad L_n^+ := P_n^+ L_n |_{\mathfrak{H}_{n,2}^+}$$

in \mathfrak{H}_n^- and \mathfrak{H}_n^+ , respectively. In analogy to (5.1), every operator L_n^- admits the form

$$L_n^- = b_n(L)^{-1/2} L_0^-(n) b_n(L)^{1/2}, \quad L_0^-(n) := P_0^-(n) L |_{\mathfrak{H}_2^-(n)}$$

and similarly for L_n^+ . The operators $L_0^\pm(n)$ are considered in $\mathfrak{H}_0^\pm(n)$, and hence they are densely defined.

Using $\mathfrak{H}_0^-(n) := P_0^-(n) \mathfrak{H}_0$ and $\mathfrak{H}_0 = (L - z_1) \mathfrak{H}_2$, $\mathfrak{H}_0^-(n)$ is the sum of sets

$$\mathfrak{H}_0^-(n) = \text{ran}(L_0^-(n) - z_1) + P_0^-(n) L \mathfrak{H}_2^+(n). \tag{5.2}$$

Thus in general the operator $L_0^-(n) - z_1$ is not surjective (unlike $L - z_1$). But the following holds.

Theorem 5.1. *Under hypothesis of Lemma 4.4 the operator $L_0^-(n) - z_1$, $n \in \mathbb{Z}$, is surjective.*

Proof. By Lemma 4.2,

$$\text{ran}(L_0^-(n) - z_1) = P_0^-(n) b_1(L) \mathfrak{H}_2^-(n) = P_0^-(n) \mathfrak{H}_0^-(n+2).$$

Now apply Lemma 4.4. □

The statement of the theorem is therefore equivalent to the statement

$$P_0^-(n) L \mathfrak{H}_2^+(n) = \{0\}. \tag{5.3}$$

Indeed, by Lemmas 4.2 and 4.4,

$$P_0^-(n) L \mathfrak{H}_2^+(n) = P_0^-(n) b_1(L) \mathfrak{H}_2^+(n) = P_0^-(n) \mathfrak{H}_0^+(n+2) = P_0^-(n) \mathfrak{H}_0^+(n) = \{0\},$$

so the sum in (5.2) implies that the operator $L_0^-(n) - z_1$ is surjective, and vice verse. In this case the operators $L_0^-(n)$ satisfy $L_0^-(n) = L_0^-(2n)$ and $L_0^-(1) = L_0^-(2n+1)$. Analogous results hold for $L_0^+(n)$ and L_n^\pm .

If L_n^{-*} is the adjoint in \mathfrak{H}_n^- of L_n^- and if $L_0^-(n)^*$ is the adjoint in $\mathfrak{H}_0^-(n)$ of $L_0^-(n)$, then the following result holds.

Lemma 5.2. $L_n^{-*} = b_n(L)^{-1/2} L_0^-(n)^* b_n(L)^{1/2}$.

Proof. The basic arguments are as in the proof of (4.1). □

Theorem 5.3. *Under hypothesis of Lemma 4.4 the operator $L_0^-(n)$, $n \in \mathbb{Z}$, is self-adjoint in $\mathfrak{H}_0^-(n)$.*

Proof. Consider the adjoint $L_0^-(n)^*$ as a linear relation in $\mathfrak{H}_0^-(n)$. Then $L_0^-(n)^*$ consists of $(y^-, x^-) \in \mathfrak{H}_0^-(n)^2$ such that $(\forall w^- \in \mathfrak{H}_2^-(n))$

$$\langle w^-, x^- \rangle_0 = \langle L_0^-(n)w^-, y^- \rangle_0.$$

Every $w^- \in \mathfrak{H}_2^-(n)$ is of the form $w^- = P_0^-(n)b_1(L)^{-1}v$ with some $v \in \mathfrak{H}_0$. Then

$$\begin{aligned} \langle L_0^-(n)w^-, y^- \rangle_0 &= \langle LP_0^-(n)b_1(L)^{-1}v, y^- \rangle_0 \\ &= \langle b_1(L)P_0^-(n)b_1(L)^{-1}v, y^- \rangle_0 + \langle P_0^-(n)b_1(L)^{-1}v, z_1y^- \rangle_0 \\ &= \langle b_1(L)P_0^-(n)b_1(L)^{-1}v, y^- \rangle_0 + \langle v, b_1(L)^{-1}z_1y^- \rangle_0. \end{aligned}$$

By applying Lemma 4.2

$$\langle b_1(L)P_0^-(n)b_1(L)^{-1}v, y^- \rangle_0 = \langle P_0^-(n+2)v, y^- \rangle_0 = \langle v, P_0^-(n+2)y^- \rangle_0.$$

On the other hand,

$$\langle w^-, x^- \rangle_0 = \langle b_1(L)^{-1}v, x^- \rangle_0 = \langle v, b_1(L)^{-1}x^- \rangle_0.$$

Therefore $(y^-, x^-) \in \mathfrak{H}_0^-(n)^2$ such that

$$b_1(L)^{-1}x^- = P_0^-(n+2)y^- + b_1(L)^{-1}z_1y^-.$$

Because $y^- = u^- + u^+$ is the sum of disjoint elements $u^\pm \in P_0^\pm(n+2)\mathfrak{H}_0^-(n)$, it follows from the above that

$$b_1(L)^{-1}x^- = u^- + b_1(L)^{-1}z_1(u^- + u^+).$$

Because $b_1(L)^{-1}\mathfrak{H}_0^-(n) = \mathfrak{H}_2^-(n-2)$ by Lemma 4.2, from here one concludes that

$$u^- \in \mathfrak{H}_2^-(n-2) \cap P_0^-(n+2)\mathfrak{H}_0^-(n) = \mathfrak{H}_2^-(n-2) \cap \mathfrak{H}_2^-(n) \cap \mathfrak{H}_2^-(n+2).$$

Sequentially

$$x^- = b_1(L)u^- + z_1(u^- + u^+) = P_0^-(n)b_1(L)u^- + z_1(u^- + u^+) = L_0^-(n)u^- + z_1u^+.$$

Finally, by applying Lemma 4.4 one gets that $u^- \in \mathfrak{H}_2^-(n)$ and $u^+ = 0$. \square

Corollary 5.4. $z_1 \in \text{res } L_0^-(n)$.

Proof. This follows from Theorems 5.1 and 5.3. \square

Under hypothesis of Lemma 4.4 and applying Lemma 5.2, the operator L_n^- is therefore self-adjoint in \mathfrak{H}_n^- . Moreover, $z_1 \in \text{res } L_n^- = \text{res } L_0^-(n)$ or, what is equivalent, $P_n^-L_n\mathfrak{H}_{n+2}^+ = \{0\}$. Similar conclusions apply to operators $L_0^+(n)$ and L_n^+ .

Lemma 5.5. *Under hypothesis of Lemma 4.4 the resolvent*

$$(L_0^-(n) - z)^{-1} = P_0^-(n)(L - z)^{-1}P_0^-(n) \quad \text{on } \mathfrak{H}_0^-(n)$$

for $z \in \text{res } L \subseteq \text{res } L_0^-(n)$ (and similarly for $L_0^+(n)$).

Proof. First we derive the resolvent formula for $z \in \text{res } L \cap \text{res } L_0^-(n)$ and then we show that $\text{res } L \subseteq \text{res } L_0^-(n)$. Consider an arbitrary $v \in \mathfrak{H}_0$. Then, for $z \in \text{res } L$, $(\exists u \in \mathfrak{H}_2) v = (L - z)u$. Projecting the latter onto $\mathfrak{H}_0^-(n)$ and applying (5.3) yields

$$P_0^-(n)v = (L_0^-(n) - z)P_0^-(n)u$$

and the resolvent formula follows for $z \in \text{res } L \cap \text{res } L_0^-(n)$.

The eigenspace

$$\mathfrak{N}_z(L_0^-(n)) = \{u^- \in \mathfrak{H}_2^-(n) \mid P_0^-(n)(L - z)u^- = 0\}$$

is nontrivial for some $z \in \mathbb{R}$ (cf. Theorem 5.3). From here and (5.3) one gets that

$$(L - z)u^- = P_0^+(n)(L - z)u^- = 0;$$

hence

$$\mathfrak{N}_z(L_0^-(n)) = \mathfrak{H}_2^-(n) \cap \mathfrak{N}_z(L).$$

If $z \notin \sigma_p(L)$ then also $z \notin \sigma_p(L_0^-(n))$, but the converse $z \notin \sigma_p(L_0^-(n))$ implies only that $\mathfrak{N}_z(L) = \mathfrak{H}_2^+(n) \cap \mathfrak{N}_z(L)$ in this case. Therefore $\sigma_p(L_0^-(n)) \subseteq \sigma_p(L)$.

Now let $z \in \text{res } L$, that is, $z \notin \sigma_p(L)$ and $\text{ran}(L - z) = \mathfrak{H}_0$. Because by (5.3)

$$\text{ran}(L - z) = \text{ran}(L_0^-(n) - z) \dot{+} \text{ran}(L_0^+(n) - z)$$

it follows that

$$\text{ran}(L_0^-(n) - z) = \mathfrak{H}_0^-(n), \quad \text{ran}(L_0^+(n) - z) = \mathfrak{H}_0^+(n)$$

so $z \in \text{res } L_0^-(n)$. □

Under the same hypothesis the resolvent of L_n^- is given by

$$(L_n^- - z)^{-1} = P_n^-(L_n - z)^{-1}P_n^- \quad \text{on } \mathfrak{H}_n^-$$

for $z \in \text{res } L_n = \text{res } L$ (and similarly for L_n^+).

We summarize the main results obtained so far in the following theorem.

Theorem 5.6. *Let $\mathfrak{H}_{n+1} \subseteq \mathfrak{H}_n$ be the scale of Hilbert spaces associated with a self-adjoint operator L in \mathfrak{H}_0 . For each $n \in \mathbb{Z}$, let P_n^- be an orthogonal projection in \mathfrak{H}_n onto a subspace $\mathfrak{H}_n^- \subseteq \mathfrak{H}_n$; \mathfrak{H}_n^+ is the orthogonal complement in \mathfrak{H}_n of \mathfrak{H}_n^- . Assume that $P_{n+1}^- \subseteq P_n^-$. Then the projections $(P_n^-)_{n \in \mathbb{Z}}$ are characterized, by scaling, by any two adjacent projections, say P_0^- and P_1^- , according to*

$$P_{2n}^- = b_n(L)^{-1}P_0^-b_n(L), \quad P_{2n+1}^- = b_n(L)^{-1}P_1^-b_n(L).$$

For each n , the subspace \mathfrak{H}_n^- (resp. \mathfrak{H}_n^+) is therefore a reducing subspace for the restriction L_n to \mathfrak{H}_{n+2} of L . The part of L_n in \mathfrak{H}_n^- (resp. \mathfrak{H}_n^+) is a self-adjoint operator.

Proof. This follows from Lemmas 4.4, 5.2, and Theorem 5.3. □

6. MIN-MAX OPERATORS IN A SUBSPACE

In the present and subsequent paragraphs $\mathfrak{M}_d^* \mathcal{G}_A = \mathcal{G}_A \mathfrak{M}_d$, as in (3.1), for an invertible Hermitian \mathcal{G}_A , and $P_{n+1}^- \subseteq P_n^-$, $n \in \mathbb{Z}$, as in Theorem 5.6. Let

$$A'_{\min} := U_A A_{\min} U_A^{-1} = \{((f^\#, \xi), (L f^\#, \mathfrak{M}_d \xi)) \mid f^\# \in \mathfrak{H}_{m+2}, \xi \in \mathbb{C}^{md}, \langle \varphi, f^\# \rangle = [\mathcal{G}_A \xi]_m\}.$$

Then A'_{\min} is a closed, densely defined, symmetric operator in \mathcal{H}'_A , whose adjoint A'^*_{\min} is given by

$$A'_{\max} := A'^*_{\min} = U_A A_{\max} U_A^{-1} = \{((f^\# + h_{m+1}(c), \xi), (L f^\# + z_1 h_{m+1}(c), \mathfrak{M}_d \xi + \eta(c))) \mid f^\# \in \mathfrak{H}_{m+2}, c \in \mathbb{C}^d, \xi \in \mathbb{C}^{md}\}.$$

If $(\mathbb{C}^d, \Gamma_0^A, \Gamma_1^A)$ is an OBT for A_{\max} then the triple $(\mathbb{C}^d, \Gamma_0'^A, \Gamma_1'^A)$, with $\Gamma_i'^A := \Gamma_i^A U_A^{-1}$, $i \in \{0, 1\}$, is an OBT for A'_{\max} .

Let

$$\Pi^\pm := P_m^\pm \oplus I_{\mathbb{C}^{md}} \quad \text{in } \mathfrak{H}_m \oplus \mathbb{C}^{md}.$$

Then Π^- (resp. Π^+) is an orthogonal (with respect to the $\mathfrak{H}_m \oplus \mathbb{C}^{md}$ -metric) projection onto a subspace $\mathfrak{H}_m^- \oplus \mathbb{C}^{md}$ (resp. $\mathfrak{H}_m^+ \oplus \mathbb{C}^{md}$). Note that

$$\Pi^- \Pi^+ \neq 0, \quad \Pi^+ \Pi^- \neq 0, \quad \Pi^- + \Pi^+ \neq I_{\mathfrak{H}_m \oplus \mathbb{C}^{md}}.$$

However, given Π^- , the above inequalities become the equalities with Π^+ replaced by the orthogonal projection $\Pi'^+ := I_{\mathfrak{H}_m \oplus \mathbb{C}^{md}} - \Pi^-$ onto

$$(\mathfrak{H}_m^- \oplus \mathbb{C}^{md})^\perp_{\mathfrak{H}_m \oplus \mathbb{C}^{md}} = \mathfrak{H}_m^+ \oplus \{0\}.$$

Likewise, given Π^+ , the above inequalities become the equalities with Π^- replaced by the orthogonal projection $\Pi'^- := I_{\mathfrak{H}_m \oplus \mathbb{C}^{md}} - \Pi^+$ onto

$$(\mathfrak{H}_m^+ \oplus \mathbb{C}^{md})^\perp_{\mathfrak{H}_m \oplus \mathbb{C}^{md}} = \mathfrak{H}_m^- \oplus \{0\}.$$

By Theorem 5.6, A'_{\min} maps

$$\text{dom } A'_{\min} \cap (\mathfrak{H}_m^- \oplus \mathbb{C}^{md}) = \Pi^- \text{dom } A'_{\min}$$

into $\mathfrak{H}_m^- \oplus \mathbb{C}^{md}$; therefore $\mathfrak{H}_m^- \oplus \mathbb{C}^{md}$ is an invariant ([28, Definition 1.7]) subspace for A'_{\min} . Let A^-_{\min} denote the part of A'_{\min} in $\mathfrak{H}_m^- \oplus \mathbb{C}^{md}$, that is

$$A^-_{\min} := A'_{\min} \upharpoonright_{\Pi^- \text{dom } A'_{\min}} = \Pi^- A'_{\min} \upharpoonright_{\Pi^- \text{dom } A'_{\min}} = \{((f^{\#-}, \xi), (L_m^- f^{\#-}, \mathfrak{M}_d \xi)) \mid f^{\#-} \in \mathfrak{H}_{m+2}^-, \xi \in \mathbb{C}^{md}, \langle \varphi, f^{\#-} \rangle = [\mathcal{G}_A \xi]_m\}.$$

Similarly one defines the part A^+_{\min} of A'_{\min} in $\mathfrak{H}_m^+ \oplus \mathbb{C}^{md}$. Because $\mathfrak{H}_m^+ \oplus \{0\}$ (resp. $\mathfrak{H}_m^- \oplus \{0\}$) is also an invariant subspace for A'_{\min} , the operator A'_{\min} is represented

by the orthogonal sum of its part A_{\min}^- in $\mathfrak{H}_m^- \oplus \mathbb{C}^{md}$ (resp. A_{\min}^+ in $\mathfrak{H}_m^+ \oplus \mathbb{C}^{md}$) and its part $L_{\min}^+ \oplus 0$ in $\mathfrak{H}_m^+ \oplus \{0\}$ (resp. $L_{\min}^- \oplus 0$ in $\mathfrak{H}_m^- \oplus \{0\}$), where the operator

$$L_{\min}^+ := L_m^+ |_{\{f^+ \in \mathfrak{H}_{m+2}^+ \mid \langle \varphi, f^+ \rangle = 0\}} \quad (\text{resp. } L_{\min}^- := L_m^- |_{\{f^- \in \mathfrak{H}_{m+2}^- \mid \langle \varphi, f^- \rangle = 0\}});$$

symbolically ($[\oplus]$ indicates both $\mathfrak{H}_m \oplus \mathbb{C}^{md}$ -orthogonal and \mathcal{H}'_A -orthogonal sum)

$$A'_{\min} = A_{\min}^- [\oplus] (L_{\min}^+ \oplus 0) = (L_{\min}^- \oplus 0) [\oplus] A_{\min}^+. \quad (6.1)$$

Let φ^- (resp. φ^+) denote the vector valued functional whose components φ_{σ}^- (resp. φ_{σ}^+) are defined by

$$\begin{aligned} \varphi_{\sigma}^- &:= b_{m+2}(L)^{1/2} P_0^-(m) b_{m+2}(L)^{-1/2} \varphi_{\sigma} \in b_{m+2}(L)^{1/2} (\mathfrak{H}_0^- \setminus \mathfrak{H}_1^-) \\ (\text{resp. } \varphi_{\sigma}^+ &:= b_{m+2}(L)^{1/2} P_0^+(m) b_{m+2}(L)^{-1/2} \varphi_{\sigma} \in b_{m+2}(L)^{1/2} (\mathfrak{H}_0^+ \setminus \mathfrak{H}_1^+)). \end{aligned}$$

The duality pairing $\langle \varphi_{\sigma}^-, \cdot \rangle$ (resp. $\langle \varphi_{\sigma}^+, \cdot \rangle$) is defined via the \mathfrak{H}_0 -scalar product in a usual way. $\langle \varphi^-, \cdot \rangle = (\langle \varphi_{\sigma}^-, \cdot \rangle): \mathfrak{H}_{m+2}^- \rightarrow \mathbb{C}^d$ denotes the action of the vector valued functional φ^- , and similarly for φ^+ .

Lemma 6.1. For $f^{\#-} \in \mathfrak{H}_{m+2}^-$

$$\langle \varphi, f^{\#-} \rangle = \langle \varphi^-, f^{\#-} \rangle = \langle h_{\sigma, m+1}^-, (L_m^- - z_1) f^{\#-} \rangle_m$$

and similarly for the action of φ on \mathfrak{H}_{m+2}^+ .

Proof. By the definition of the duality pairing and that of φ_{σ}^- ,

$$\begin{aligned} \langle \varphi_{\sigma}^-, f^{\#-} \rangle &= \langle b_{m+2}(L)^{-1/2} \varphi_{\sigma}^-, b_{m+2}(L)^{1/2} f^{\#-} \rangle_0 \\ &= \langle P_0^-(m) b_{m+2}(L)^{-1/2} \varphi_{\sigma}, b_{m+2}(L)^{1/2} f^{\#-} \rangle_0 \\ &= \langle b_{m+2}(L)^{-1/2} \varphi_{\sigma}, P_0^-(m) b_{m+2}(L)^{1/2} f^{\#-} \rangle_0. \end{aligned}$$

But

$$b_{m+2}(L)^{1/2} f^{\#-} \in b_{m+2}(L)^{1/2} \mathfrak{H}_{m+2}^- = b_{m+2}(L)^{1/2} P_{m+2}^- \mathfrak{H}_{m+2} = \mathfrak{H}_0^-(m+2)$$

and hence by Lemma 4.4 $b_{m+2}(L)^{1/2} f^{\#-} \in \mathfrak{H}_0^-(m)$; therefore

$$\begin{aligned} \langle b_{m+2}(L)^{-1/2} \varphi_{\sigma}, P_0^-(m) b_{m+2}(L)^{1/2} f^{\#-} \rangle_0 &= \langle b_{m+2}(L)^{-1/2} \varphi_{\sigma}, b_{m+2}(L)^{1/2} f^{\#-} \rangle_0 \\ &= \langle \varphi_{\sigma}, f^{\#-} \rangle. \end{aligned}$$

This proves the first equality. Using that $b_{m+2}(L)^{1/2} f^{\#-} \in \mathfrak{H}_0^-(m)$, the second equality is due to

$$\begin{aligned} \langle h_{\sigma, m+1}^-, (L_m^- - z_1) f^{\#-} \rangle_m &= \langle h_{\sigma, m+1}^-, b_1(L) f^{\#-} \rangle_m \\ &= \langle b_m(L)^{1/2} P_m^- h_{\sigma, m+1}, b_{m+2}(L)^{1/2} f^{\#-} \rangle_0 \\ &= \langle P_0^-(m) b_{m+2}(L)^{-1/2} \varphi_{\sigma}, b_{m+2}(L)^{1/2} f^{\#-} \rangle_0 \\ &= \langle b_{m+2}(L)^{-1/2} \varphi_{\sigma}, b_{m+2}(L)^{1/2} f^{\#-} \rangle_0 = \langle \varphi_{\sigma}, f^{\#-} \rangle. \end{aligned}$$

The proof of $\langle \varphi, \cdot \rangle$ on \mathfrak{H}_{m+2}^+ is analogous. □

By the lemma the boundary conditions defining the operators L_{\min}^{\pm} are therefore reduced to $\langle \varphi^{\pm}, f^{\pm} \rangle = 0$, $f^{\pm} \in \mathfrak{H}_{m+2}$, where $\varphi^{-} + \varphi^{+} = \varphi$. Explicitly

$$L_{\min}^{-} := L_m^{-} |_{\{f^{-} \in \mathfrak{H}_{m+2}^{-} | \langle \varphi^{-}, f^{-} \rangle = 0\}}, \quad L_{\min}^{+} := L_m^{+} |_{\{f^{+} \in \mathfrak{H}_{m+2}^{+} | \langle \varphi^{+}, f^{+} \rangle = 0\}}.$$

Just like the functionals φ_{σ} define the elements $h_{\sigma j} := b_j(L)^{-1} \varphi_{\sigma}$, $j \in J$, that generate the linear space \mathfrak{K}_A , the functionals φ_{σ}^{\pm} define the elements

$$h_{\sigma j}^{\pm} := b_j(L)^{-1} \varphi_{\sigma}^{\pm} = P_{-m-2+2j}^{\pm} h_{\sigma j} \tag{6.2}$$

that generate (span) the linear subspaces \mathfrak{K}_A^{\pm} of \mathfrak{K}_A ; that is, $\mathfrak{K}_A = \mathfrak{K}_A^{-} \dot{+} \mathfrak{K}_A^{+}$. The proof of the second equality in (6.2) uses the definition of $P_0^{\pm}(\cdot)$ and then Lemma 4.4, in the same spirit as in the proof of Lemma 6.1.

Unlike the case of A'_{\min} , the operator A'_{\max} does not commute with the projection Π^{-} (resp. Π^{+}). The reason is that now the projection of $h_{m+1}(c)$ onto \mathfrak{H}_m^{-} affects the value of the extra term $\eta(c) \in \mathbb{C}^{md}$. This seems to be better seen in the representation of the operator A'_{\max} in the space $\mathfrak{H}_m \dot{+} \mathfrak{K}_A$, i.e. in analyzing the operator A_{\max} . Thus we have by Lemma 3.1 (here $k \in \mathfrak{K}_A$)

$$\begin{aligned} A_{\max}(f^{\#} + h_{m+1}(c) + k) &= L_{m-2}(f^{\#} + h_{m+1}(c)) + k', \\ k' \in \mathfrak{K}_A, \quad d(k') &= \mathfrak{M}_d d(k) \end{aligned}$$

and

$$L_{m-2} h_{m+1}(c) = z_1 h_{m+1}(c) + h_m(c),$$

where

$$h_m(c) = b_1(L) h_{m+1}(c) = \sum_{\alpha} [\eta(c)]_{\alpha} h_{\alpha} = \sum_{\sigma} c_{\sigma} h_{\sigma m} \in \mathfrak{K}_A.$$

Now projecting $f^{\#} + h_{m+1}(c) + k$ onto $\mathfrak{H}_m^{-} \dot{+} \mathfrak{K}_A$ one gets that

$$\begin{aligned} A_{\max} U_A^{-1} \Pi^{-} U_A (f^{\#} + h_{m+1}(c) + k) &= L_{m-2}^{-} (f^{\# -} + h_{m+1}^{-}(c)) + k' \\ &= L_m^{-} f^{\# -} + z_1 h_{m+1}^{-}(c) + k' + h_m^{-}(c) \end{aligned}$$

with

$$h_m^{-}(c) := b_1(L) h_{m+1}^{-}(c) = \sum_{\alpha} [\eta(c)]_{\alpha} h_{\alpha}^{-} = \sum_{\sigma} c_{\sigma} h_{\sigma m}^{-} \in \mathfrak{K}_A^{-}$$

(it is precisely for this reason why $\eta(c)$ changes to $\eta^{-}(c) \neq \eta(c)$; see below), while

$$\begin{aligned} U_A^{-1} \Pi^{-} U_A A_{\max} (f^{\#} + h_{m+1}(c) + k) &= L_m^{-} f^{\# -} + z_1 h_{m+1}^{-}(c) + k' + h_m(c) \\ &= A_{\max} U_A^{-1} \Pi^{-} U_A (f^{\#} + h_{m+1}(c) + k) + h_m^{+}(c) \end{aligned}$$

with $h_m^{+}(c) \in \mathfrak{K}_A^{+}$ defined similarly as $h_m^{-}(c)$. Because $h_m^{\pm}(c) \in \mathfrak{K}_A^{\pm}$ and $\mathfrak{K}_A^{\pm} \subseteq \mathfrak{K}_A$, it follows that

$$h_m^{\pm}(c) = \sum_{\alpha} [\eta(c)]_{\alpha} h_{\alpha}^{\pm} = \sum_{\alpha} [\eta^{\pm}(c)]_{\alpha} h_{\alpha}$$

for $\eta^\pm(c) \in \mathbb{C}^{md}$ given by

$$\eta^\pm(c) := \tilde{\mathcal{G}}_A^{-1} \langle h, h_m^\pm(c) \rangle_{-m} = \tilde{\mathcal{G}}_A^{-1} \tilde{\mathcal{G}}_A^\pm \eta(c)$$

with the matrix

$$\tilde{\mathcal{G}}_A^\pm = ([\tilde{\mathcal{G}}_A^\pm]_{\alpha\alpha'}), \quad [\tilde{\mathcal{G}}_A^\pm]_{\alpha\alpha'} := \langle h_\alpha, h_{\alpha'}^\pm \rangle_{-m}.$$

With this notation, and going back to the representation of A_{\max} in $\mathfrak{H}_m \oplus \mathbb{C}^{md}$, one gets that

$$A'_{\max} \Pi^-(f^\# + h_{m+1}(c), \xi) = (L_m^- f^{\#-} + z_1 h_{m+1}^-(c), \mathfrak{M}_d \xi + \eta^-(c))$$

while

$$\Pi^- A'_{\max}(f^\# + h_{m+1}(c), \xi) = (L_m^- f^{\#-} + z_1 h_{m+1}^-(c), \mathfrak{M}_d \xi + \eta(c)).$$

Similarly, projecting $(f^\# + h_{m+1}(c), \xi)$ onto $\mathfrak{H}_m^+ \oplus \{0\}$ gives

$$A'_{\max} \Pi^+(f^\# + h_{m+1}(c), \xi) = (L_m^+ f^{\#+} + z_1 h_{m+1}^+(c), \eta^+(c))$$

while

$$\Pi^+ A'_{\max}(f^\# + h_{m+1}(c), \xi) = (L_m^+ f^{\#+} + z_1 h_{m+1}^+(c), 0).$$

From these formulas one observes that one still is able to represent the extension of the operator A'_{\max} (but not the operator A'_{\max} itself) as the orthogonal sum of its parts in subspaces $\mathfrak{H}_m^- \oplus \mathbb{C}^{md}$ (resp. $\mathfrak{H}_m^+ \oplus \mathbb{C}^{md}$) and $\mathfrak{H}_m^+ \oplus \{0\}$ (resp. $\mathfrak{H}_m^- \oplus \{0\}$), similarly as in (6.1), by moving an element $(0, \eta^+(c))$ from $A'_{\max} \Pi^+$ to $A'_{\max} \Pi^-$.

To make this precise, one therefore introduces the linear relation

$$A_{\max}^- := \{((f^{\#-} + h_{m+1}^-(c), \xi), (L_m^- f^{\#-} + z_1 h_{m+1}^-(c), \mathfrak{M}_d \xi + \eta(c))) \mid f^{\#-} \in \mathfrak{H}_{m+2}^-, c \in \mathbb{C}^d, \xi \in \mathbb{C}^{md}\}$$

in $\mathfrak{H}_m^- \oplus \mathbb{C}^{md}$ with the multivalued part

$$\text{mul } A_{\max}^- = \{0\} \times \eta^+(\Sigma^-), \quad \Sigma^- := \left\{ c \in \mathbb{C}^d \mid \sum_{\sigma} c_{\sigma} \varphi_{\sigma}^- = 0 \right\}$$

(the multivalued part is exactly the orthogonal complement in \mathcal{H}'_A of $\text{dom } A_{\min}^-$) and the operator

$$L_{\max}^- \oplus 0 = \Pi'^- A'_{\max} \mid_{\Pi'^- \text{ dom } A'_{\max}}$$

in $\mathfrak{H}_m^- \oplus \{0\}$ with

$$L_{\max}^- := \{(f^{\#-} + h_{m+1}^-(c), L_m^- f^{\#-} + z_1 h_{m+1}^-(c)) \mid f^{\#-} \in \mathfrak{H}_{m+2}^-, c \in \mathbb{C}^d\}.$$

Analogously one defines the linear relation A_{\max}^+ in $\mathfrak{H}_m^+ \oplus \mathbb{C}^{md}$, with the multivalued part $\{0\} \times \eta^-(\Sigma^+)$, and the operator L_{\max}^+ in \mathfrak{H}_m^+ . Note that the domain of the operator L_{\max}^- in \mathfrak{H}_m^- can be also written thus

$$\text{dom } L_{\max}^- = \mathfrak{H}_{m+2}^- \dot{+} \mathfrak{N}_z(L_{\max}^-), \quad z \in \text{res } L_m^-$$

with the eigenspace

$$\mathfrak{N}_z(L_{\max}^-) = (L_m^- - z_1)(L_m^- - z)^{-1}h_{m+1}^-(\mathbb{C}^d)$$

and similarly for L_{\max}^+ . (The operators L_{\max}^\pm should not be confused with the triplet adjoint L_{\max} ; as we show below, L_{\max}^- is the adjoint in \mathfrak{H}_m^- of L_{\min}^- , and similarly for L_{\max}^+ .)

It follows from the above constructions that the orthogonal (both in $\mathfrak{H}_m \oplus \mathbb{C}^{md}$ -metric and in \mathcal{H}'_A -metric) componentwise sum of linear relations (cf. [18, 19, 22] for the notation)

$$A_{\max}^-[\widehat{\oplus}](L_{\max}^+ \oplus 0) = (L_{\max}^- \oplus 0)[\widehat{\oplus}]A_{\max}^+ \tag{6.3}$$

is an extension in $\mathfrak{H}_m \oplus \mathbb{C}^{md}$ of the operator A'_{\max} . By comparing (6.1) with (6.3) one concludes that $A_{\min}^- \subseteq A_{\max}^-$ and $L_{\min}^- \subseteq L_{\max}^-$, and similarly for A_{\min}^+ and L_{\min}^+ . In fact, one can say more.

Theorem 6.2. *The linear relation $A_{\max}^- = A_{\min}^{-*}$ is the adjoint in \mathcal{H}'_A of a nondensely defined (in general), closed, symmetric operator A_{\min}^- .*

Proof. The main arguments are as in the proof of the self-adjointness of L_m^- (Theorem 5.3) by using in addition that the boundary condition for $(f^{\#-}, \xi) \in \text{dom } A_{\min}^-$ implies that $(\forall c \in \mathbb{C}^d)$

$$\langle w, b_m(L)^{1/2}h_{m+1}^-(c) \rangle_0 = \langle \xi, \mathcal{G}_A \eta(c) \rangle_{\mathbb{C}^{md}}, \quad f^{\#-} = b_{m+2}(L)^{-1/2}P_0^-(m)w, \tag{6.4}$$

$w \in \mathfrak{H}_0$; note that

$$b_m(L)^{1/2}h_{m+1}^-(c) = b_{m+2}(L)^{-1/2} \sum_{\sigma} c_{\sigma} \varphi_{\sigma}^-$$

and the representation of $f^{\#-}$ is shown in the proof of Lemma 6.1. The duality pairing then reads

$$\begin{aligned} \langle \varphi^-, f^{\#-} \rangle &= \langle b_{m+2}(L)^{-1/2} \varphi^-, b_{m+2}(L)^{1/2} f^{\#-} \rangle_0 \\ &= \langle b_{m+2}(L)^{-1/2} \varphi^-, P_0^-(m)w \rangle_0, \end{aligned}$$

but $b_{m+2}(L)^{-1/2} \varphi^- \in \mathfrak{H}_0^-(m)$, so the boundary condition reads

$$\langle \varphi^-, f^{\#-} \rangle = \langle b_{m+2}(L)^{-1/2} \varphi^-, w \rangle_0 = [\mathcal{G}_A \xi]_m$$

from which (6.4) follows.

Now one computes A_{\min}^{-*} ; as a linear relation, it is the set of $((y^-, \xi_y), (x^-, \xi_x)) \in (\mathfrak{H}_m^- \oplus \mathbb{C}^{md})^2$ such that $(\forall (f^{\#-}, \xi) \in \text{dom } A_{\min}^-)$

$$\langle f^{\#-}, x^- \rangle_m + \langle \xi, \mathcal{G}_A \xi_x \rangle_{\mathbb{C}^{md}} = \langle L_m^- f^{\#-}, y^- \rangle_m + \langle \mathfrak{M}_d \xi, \mathcal{G}_A \xi_y \rangle_{\mathbb{C}^{md}}. \tag{6.5}$$

Applying the representation

$$\begin{aligned} x^- &= b_m(L)^{-1/2}u^-, \quad u^- \in \mathfrak{H}_0^-(m), \\ y^- &= b_m(L)^{-1/2}v^-, \quad v^- \in \mathfrak{H}_0^-(m) \end{aligned}$$

and using that $b_1(L)^{-1}\mathfrak{H}_0^-(m) = \mathfrak{H}_2^-(m)$ one gets that

$$\langle f^{\#-}, x^- \rangle_m = \langle w, b_1(L)^{-1}u^- \rangle_0$$

and

$$\begin{aligned} \langle L_m^- f^{\#-}, y^- \rangle_m &= \langle b_1(L)f^{\#-}, y^- \rangle_m + \langle f^{\#-}, z_1 y^- \rangle_m \\ &= \langle w, v^- \rangle_0 + \langle w, b_1(L)^{-1}z_1 v^- \rangle_0. \end{aligned}$$

Therefore (6.5) reads

$$\langle w, v^- - b_1(L)^{-1}(u^- - z_1 v^-) \rangle_0 = \langle \xi, \mathcal{G}_A(\xi_x - \mathfrak{M}_d \xi_y) \rangle_{\mathbb{C}^{md}}.$$

Comparing the latter with (6.4) yields

$$\begin{aligned} v^- - b_1(L)^{-1}(u^- - z_1 v^-) &= b_m(L)^{1/2} h_{m+1}^-(c), \\ \xi_x &= \mathfrak{M}_d \xi_y + \eta(c). \end{aligned}$$

The first equation above implies that

$$v^- - b_m(L)^{1/2} h_{m+1}^-(c) \in b_1(L)^{-1}\mathfrak{H}_0^-(m) = \mathfrak{H}_2^-(m),$$

that is,

$$y^- = f^- + h_{m+1}^-(c), \quad f^- \in \mathfrak{H}_{m+2}^-.$$

Then

$$x^- = z_1 y^- + b_1(L)f^- = L_m^- f^- + z_1 h_{m+1}^-(c).$$

This proves $A_{\min}^- = A_{\max}^-$. It remains to verify that A_{\min}^- is closed. The adjoint A_{\max}^{-*} consists of $((y^-, \xi_y), (x^-, \xi_x)) \in (\mathfrak{H}_m^- \oplus \mathbb{C}^{md})^2$ such that $(\forall f^{\#-} \in \mathfrak{H}_{m+2}^- \forall c \in \mathbb{C}^d \forall \xi \in \mathbb{C}^{md})$

$$\begin{aligned} \langle f^{\#-} + h_{m+1}^-(c), x^- \rangle_m + \langle \xi, \mathcal{G}_A \xi_x \rangle_{\mathbb{C}^{md}} &= \langle L_m^- f^{\#-} + z_1 h_{m+1}^-(c), y^- \rangle_m \\ &\quad + \langle \mathfrak{M}_d \xi + \eta(c), \mathcal{G}_A \xi_y \rangle_{\mathbb{C}^{md}}. \end{aligned}$$

Using the representation of $f^{\#-}, x^-, y^-$ as above, and noting that

$$\langle h_{m+1}^-(c), x^- \rangle_m = \langle c, \langle h_{m+1}^-, x^- \rangle_m \rangle_{\mathbb{C}^d}, \quad \langle \eta(c), \mathcal{G}_A \xi_y \rangle_{\mathbb{C}^{md}} = \langle c, [\mathcal{G}_A \xi_y]_m \rangle_{\mathbb{C}^d}$$

one gets that

$$\begin{aligned} 0 &= \langle w, v^- - b_1(L)^{-1}(u^- - z_1 v^-) \rangle_0 + \langle c, \langle h_{m+1}^-, z_1 y^- - x^- \rangle_m + [\mathcal{G}_A \xi_y]_m \rangle_{\mathbb{C}^d} \\ &\quad + \langle \xi, \mathcal{G}_A(\mathfrak{M}_d \xi_y - \xi_x) \rangle_{\mathbb{C}^{md}} \end{aligned}$$

and from which one concludes that

$$v^- = b_1(L)^{-1}(u^- - z_1 v^-) \in \mathfrak{H}_2^-(m) \Rightarrow x^- = L_m^- y^-, \quad y^- \in \mathfrak{H}_{m+2}^-$$

and

$$\langle h_{m+1}^-, x^- - z_1 y^- \rangle_m = \langle h_{m+1}^-, (L_m^- - z_1)y^- \rangle_m = \langle \varphi, y^- \rangle = [\mathcal{G}_A \xi_y]_m$$

(cf. Lemma 6.1) and $\xi_x = \mathfrak{M}_d \xi_y$. Thus A_{\min}^- is closed, and this completes the proof. \square

The above proof also shows that:

Corollary 6.3. *The operator $L_{\max}^- = L_{\min}^{-*}$ is the adjoint in \mathfrak{H}_m^- of a densely defined, closed, symmetric operator L_{\min}^- .*

From here one concludes that L_{\min}^- (resp. L_{\min}^+) is an essentially self-adjoint operator in \mathfrak{H}_0^- (resp. \mathfrak{H}_0^+). Since A_{\min}^- extends L_{\min}^- to \mathcal{H}'_A^- just like A_{\min} extends L_{\min} to \mathcal{H}_A it is therefore a subject of interest to formulate a similar realization theorem in the A-model for the symmetric operator L_{\min}^- . This is done in the next (the last) paragraph.

7. REALIZATION THEOREM IN A SUBSPACE

By a straightforward computation and applying Lemma 6.1, the boundary form of the linear relation A_{\max}^- is given by

$$\begin{aligned} & [(f^{\#-} + h_{m+1}^-(c), \xi), (L_m^- g^{\#-} + z_1 h_{m+1}^-(c'), \mathfrak{M}_d \xi' + \eta(c'))]'_A \\ & - [(L_m^- f^{\#-} + z_1 h_{m+1}^-(c), \mathfrak{M}_d \xi + \eta(c)), (g^{\#-} + h_{m+1}^-(c'), \xi')]'_A \\ & = \langle c, \langle \varphi^-, g^{\#-} \rangle - [\mathcal{G}_A \xi']_m \rangle_{\mathbb{C}^d} - \langle \langle \varphi^-, f^{\#-} \rangle - [\mathcal{G}_A \xi]_m, c' \rangle_{\mathbb{C}^d} \end{aligned}$$

for $f^{\#-}, g^{\#-} \in \mathfrak{H}_{m+2}^-$; $c, c' \in \mathbb{C}^d$; $\xi, \xi' \in \mathbb{C}^{md}$. By introducing the mappings from A_{\max}^- to \mathbb{C}^d by

$$\Gamma_0^A - \widehat{f}^- := c, \quad \Gamma_1^A - \widehat{f}^- := \langle \varphi^-, f^{\#-} \rangle - [\mathcal{G}_A \xi]_m, \quad (7.1)$$

$$\widehat{f}^- = ((f^{\#-} + h_{m+1}^-(c), \xi), (L_m^- f^{\#-} + z_1 h_{m+1}^-(c), \mathfrak{M}_d \xi + \eta(c))) \in A_{\max}^-$$

the above boundary form simplifies thus

$$\begin{aligned} [f^-, g'^-]'_A - [f'^-, g^-]'_A &= \langle \Gamma_0^A - \widehat{f}^-, \Gamma_1^A - \widehat{g}^- \rangle_{\mathbb{C}^d} - \langle \Gamma_1^A - \widehat{f}^-, \Gamma_0^A - \widehat{g}^- \rangle_{\mathbb{C}^d}, \\ \widehat{f}^- &= (f^-, f'^-) \in A_{\max}^-, \quad \widehat{g}^- = (g^-, g'^-) \in A_{\max}^- \end{aligned}$$

and it therefore represents the Green identity. Consider $\Gamma^A -: \widehat{f}^- \mapsto (\Gamma_0^A - \widehat{f}^-, \Gamma_1^A - \widehat{f}^-)$ from A_{\max}^- to $\mathbb{C}^d \times \mathbb{C}^d$ as an (isometric) linear relation from $(\mathcal{H}'_A^-)^2$ to $\mathbb{C}^d \times \mathbb{C}^d$. Thus by definition $\text{dom } \Gamma^A - = A_{\max}^-$ and $\text{ker } \Gamma^A - = A_{\min}^-$. Moreover, the multivalued part $\text{mul } \Gamma^A -$ consists of $(c, 0)$ such that $c \in \Sigma^- \cap \Sigma^+ = \{0\}$; hence $\Gamma^A -$ is an operator. Below we show that $\Gamma^A -$ is a unitary relation from $(\mathcal{H}'_A^-)^2$ to $\mathbb{C}^d \times \mathbb{C}^d$ (by the above, it would actually suffice to show that $\text{dom}(\Gamma^A -)^{[+]} = \text{ran } \Gamma^A -$). By [12, Corollary 2.4(iii)] this would imply that $\Gamma^A -$ is surjective, and that therefore the triple $(\mathbb{C}^d, \Gamma_0^A - , \Gamma_1^A -)$ is an OBT for A_{\max}^- .

Lemma 7.1. *$(\mathbb{C}^d, \Gamma_0^A - , \Gamma_1^A -)$ is an OBT for A_{\max}^- .*

Proof. By definition, the Krein space adjoint $(\Gamma^A -)^{[+]}$ is a linear relation consisting of

$$((\chi, \chi'), ((y^-, \xi_y), (x^-, \xi_x))) \in \mathbb{C}^{2d} \times (\mathfrak{H}_m^- \oplus \mathbb{C}^{md})^2$$

such that $(\forall f^{\#-} \in \mathfrak{H}_{m+2}^- \forall c \in \mathbb{C}^d \forall \xi \in \mathbb{C}^{md})$

$$\begin{aligned} & \langle f^{\#-} + h_{m+1}^-(c), x^- \rangle_m + \langle \xi, \mathcal{G}_A \xi_x \rangle_{\mathbb{C}^{md}} \\ & - \langle L_m^- f^{\#-} + z_1 h_{m+1}^-(c), y^- \rangle_m - \langle \mathfrak{M}_d \xi + \eta(c), \mathcal{G}_A \xi_y \rangle_{\mathbb{C}^{md}} \\ & = \langle c, \chi' \rangle_{\mathbb{C}^d} - \langle \langle h_{m+1}^-, (L_m^- - z_1) f^{\#-} \rangle_m - [\mathcal{G}_A \xi]_m, \chi \rangle_{\mathbb{C}^d}. \end{aligned}$$

The above equation splits into three equations

$$\begin{aligned} (\forall f^{\#-}) \quad & \langle f^{\#-}, x^- - z_1 h_{m+1}^-(\chi) \rangle_m = \langle L_m^- f^{\#-}, y^- - h_{m+1}^-(\chi) \rangle_m, \\ (\forall c) \quad & 0 = \langle c, \langle h_{m+1}^-, x^- - z_1 y^- \rangle_m - [\mathcal{G}_A \xi_y]_m - \chi' \rangle_{\mathbb{C}^d}, \\ (\forall \xi) \quad & 0 = \langle \xi, \mathcal{G}_A (\xi_x - \mathfrak{M}_d \xi_y - \eta(\chi)) \rangle_{\mathbb{C}^{md}}. \end{aligned}$$

Because L_m^- is self-adjoint in \mathfrak{H}_m^- , the first equation gives

$$y^- = f^- + h_{m+1}^-(\chi), \quad f^- \in \mathfrak{H}_{m+2}^-, \quad x^- = L_m^- f^- + z_1 h_{m+1}^-(\chi).$$

Then the second equation yields

$$\chi' = \langle \varphi^-, f^- \rangle - [\mathcal{G}_A \xi_y]_m \quad (\text{Lemma 6.1}).$$

Finally, by the third equation

$$\xi_x = \mathfrak{M}_d \xi_y + \eta(\chi).$$

As a result $(\Gamma^{A-})^{[+]} = (\Gamma^{A-})^{-1}$. □

Let

$$\Gamma_0^-(f^{\#-} + h_{m+1}^-(c)) := c, \quad \Gamma_1^-(f^{\#-} + h_{m+1}^-(c)) := \langle \varphi^-, f^{\#-} \rangle \quad (7.2)$$

for $f^{\#-} + h_{m+1}^-(c) \in \text{dom } L_{\max}^-$. The above proof also shows that:

Corollary 7.2. $(\mathbb{C}^d, \Gamma_0^-, \Gamma_1^-)$ is an OBT for L_{\max}^- .

We are now ready to state the main realization theorem in the A-model for the symmetric operator L_{\min}^- , by assuming (3.1) and $P_{n+1}^- \subseteq P_n^-$, $n \in \mathbb{Z}$.

Theorem 7.3. *The extensions to \mathcal{H}'_A^- of a densely defined, closed, symmetric operator $L_{\min}^- = L_{\min} \cap (\mathfrak{H}_m^-)^2$ in \mathfrak{H}_m^- , which has defect numbers (d, d) and which is essentially self-adjoint in \mathfrak{H}_0^- , are described by the proper extensions in \mathcal{H}'_A^- of a nondensely defined (in general), closed, symmetric operator $A_{\min}^- = A'_{\min} \cap (\mathfrak{H}_m^- \oplus \mathbb{C}^d)^2$. A proper extension A_{Θ}^- is characterized by restricting the adjoint linear relation $A_{\max}^- = A_{\min}^{-*}$ in \mathcal{H}'_A^- to the set of $\widehat{f}^- \in A_{\max}^-$ such that the pair $(\Gamma_0^A \widehat{f}^-, \Gamma_1^A \widehat{f}^-)$ is an element of a linear relation Θ in \mathbb{C}^d ; an OBT $(\mathbb{C}^d, \Gamma_0^A, \Gamma_1^A)$ for A_{\max}^- is as in (7.1). The Krein–Naimark resolvent formula for a (closed) proper extension A_{Θ}^- reads*

$$(A_{\Theta}^- - z)^{-1} = (A_0^- - z)^{-1} + \gamma_A^-(z)(\Theta - M_A^-(z))^{-1} \gamma_A^-(\bar{z})^*$$

for $z \in \text{res } A_0^- \cap \text{res } A_\Theta^-$. A distinguished self-adjoint extension A_0^- of A_{\min}^- is a self-adjoint operator $A_0^- := A_{\{0\} \times \mathbb{C}^d}^-$ whose resolvent is given by

$$(A_0^- - z)^{-1} = (L_m^- - z)^{-1} \oplus (\mathfrak{M}_d - z)^{-1}$$

for $z \in \text{res } A_0^- = \text{res } L_m^- \setminus \{z_1\}$. The γ -field γ_A^- and the Weyl function M_A^- corresponding to $(\mathbb{C}^d, \Gamma_0^{A^-}, \Gamma_1^{A^-})$ are described by

$$\gamma_A^-(z) = ((L_m^- - z_1)(L_m^- - z)^{-1}h_{m+1}^-(\cdot), -(\mathfrak{M}_d - z)^{-1}\eta(\cdot)) \quad \text{on } \mathbb{C}^d,$$

$$M_A^-(z) = q^-(z) + r(z) \quad \text{on } \mathbb{C}^d$$

for $z \in \text{res } A_0^-$. The matrix valued function q^- given by

$$q^-(z) = ([q^-(z)]_{\sigma\sigma'}) \in [\mathbb{C}^d], \quad z \in \text{res } L_m^-, \\ [q^-(z)]_{\sigma\sigma'} := (z - z_1) \langle \varphi_\sigma^-, (L_m^- - z)^{-1}h_{\sigma',m+1}^- \rangle$$

is the Weyl function which corresponds to the OBT $(\mathbb{C}^d, \Gamma_0^-, \Gamma_1^-)$, (7.2), for the adjoint operator $L_{\max}^- = L_{\min}^{*-}$ in \mathfrak{H}_m^- .

Proof. In view of what has been achieved so far, it remains to compute the γ -field and the Weyl function. But these functions follow straightforwardly from their definitions as long as one notices that the eigenspace of A_{\max}^- for the eigenvalue $z \in \text{res } L_m^- \setminus \{z_1\}$ consists of $(f^{\#-} + h_{m+1}^-(c), \xi) \in \text{dom } A_{\max}^-$ such that

$$f^{\#-} = (z - z_1)(L_m^- - z)^{-1}h_{m+1}^-(c), \quad \xi = -(\mathfrak{M}_d - z)^{-1}\eta(c).$$

Because $L_{\max}^- = A_{\max}^- \cap (\mathfrak{H}_m^- \oplus \{0\})^2$, the results for L_{\max}^- are derived analogously. \square

In particular, putting $P_n^- = I_{\mathfrak{H}_n}$ (hence $P_n^+ = 0$), $n \in \mathbb{Z}$, the part of the theorem concerning the Weyl function q^- yields the following:

Corollary 7.4. *The Krein Q -function q is the Weyl function associated with the OBT $(\mathbb{C}^d, \Gamma_0, \Gamma_1)$,*

$$\Gamma_0(f^\# + h_{m+1}(c)) := c, \quad \Gamma_1(f^\# + h_{m+1}(c)) := \langle \varphi, f^\# \rangle$$

$(f^\# \in \mathfrak{H}_{m+2}, c \in \mathbb{C}^d)$, for the adjoint L_{\min}^* of L_{\min} in \mathfrak{H}_m . The domain $\text{dom } L_{\min}^* = \mathfrak{H}_{m+2} \dot{+} \mathfrak{N}_z(L_{\min}^*)$, where the eigenspace $\mathfrak{N}_z(L_{\min}^*) = (L - z)^{-1}h_m(\mathbb{C}^d)$, $z \in \text{res } L$. \square

An analogous theorem can be formulated for L_{\min}^+ as well, where the corresponding Weyl function $M_A^+ = q^+ + r$ is the sum of the Weyl function q^+ of L_{\min}^+ and the generalized Nevanlinna function r .

Let

$$\widehat{h}_\sigma := b_{m+2}(L)^{-1/2}\varphi_\sigma \in \mathfrak{H}_0 \setminus \mathfrak{H}_1.$$

Using this definition and the operator identity

$$(L - z_1)(L - z)^{-1} = I_{\mathfrak{H}_0} + (z - z_1)(L - z)^{-1}$$

the Weyl function q is rewritten in terms of the initial operator L and the reference \mathfrak{H}_0 -scalar product according to

$$[q(z)]_{\sigma\sigma'} = (z - z_1) \langle \widehat{h}_\sigma, \widehat{h}_{\sigma'} \rangle_0 + (z - z_1)^2 \langle \widehat{h}_\sigma, (L - z)^{-1} \widehat{h}_{\sigma'} \rangle_0,$$

$z \in \text{res } L$. Using in addition (5.3) and applying [28, Proposition 5.26] and Lemma 5.5, the Weyl function q^- admits the form

$$[q^-(z)]_{\sigma\sigma'} = (z - z_1) \langle \widehat{h}_\sigma, P_0^-(m) \widehat{h}_{\sigma'} \rangle_0 + (z - z_1)^2 \langle \widehat{h}_\sigma, P_0^-(m)(L - z)^{-1} \widehat{h}_{\sigma'} \rangle_0,$$

$z \in \text{res } L$, and similarly for q^+ . Thus the Weyl function $q = q^- + q^+$ of the symmetric operator L_{\min} is the sum of the Weyl functions q^\pm of the corresponding symmetric restrictions L_{\min}^\pm . The latter property of additivity is clearly a consequence of the initial hypothesis that the subspaces \mathfrak{H}_0^\pm reduce the operator L (Theorem 5.6).


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