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Article

Universality in Short Intervals of the Riemann Zeta-Function Twisted by Non-Trivial Zeros

Antanas Laurinčikas [1,†] and Darius Šiaučiūnas [2,*,†]

[1] Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko str. 24, LT-03225 Vilnius, Lithuania; antanas.laurincikas@mif.vu.lt
[2] Regional Development Institute, Šiauliai University, P. Višinskio str. 25, LT-76351 Šiauliai, Lithuania
* Correspondence: darius.siauciunas@su.lt
† These authors contributed equally to this work.

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Abstract: Let $0 < \gamma_1 < \gamma_2 < \cdots \leqslant \gamma_k \leqslant \cdots$ be the sequence of imaginary parts of non-trivial zeros of the Riemann zeta-function $\zeta(s)$. Using a certain estimate on the pair correlation of the sequence $\{\gamma_k\}$ in the intervals $[N, N+M]$ with $N^{1/2+\varepsilon} \leqslant M \leqslant N$, we prove that the set of shifts $\zeta(s + ih\gamma_k)$, $h > 0$, approximating any non-vanishing analytic function defined in the strip $\{s \in \mathbb{C} : 1/2 < \mathrm{Re}s < 1\}$ with accuracy $\varepsilon > 0$ has a positive lower density in $[N, N+M]$ as $N \to \infty$. Moreover, this set has a positive density for all but at most countably $\varepsilon > 0$. The above approximation property remains valid for certain compositions $F(\zeta(s))$.

Keywords: Montgomery pair correlation conjecture; non-trivial zeros; Riemann zeta-function; universality

MSC: 11M06; 11M26

1. Introduction

The Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, is defined, for $\sigma > 1$, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the infinite product is taken over all prime numbers, and has analytic continuation over the whole complex plane, except for the point $s = 1$ which is a simple pole with residue 1. The function $\zeta(s)$ and its value distribution play an important role not only in analytic number theory but in mathematics in general.

It is well known by a Bohr and Courant work [1] that the set of values of $\zeta(\sigma + it)$ with any fixed $\sigma \in (1/2, 1]$ is dense in $\mathbb{C}$. Voronin obtained [2] the infinite-dimensional version of the Bohr–Courant theorem, proving the so-called universality of $\zeta(s)$. This means that every non-vanishing analytic function in the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ can be approximated by shifts $\zeta(s + i\tau)$. We recall the modern version of the Voronin theorem. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then, for $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and every $\varepsilon > 0$, the inequality

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0$$

is true; see, for example, [3–6]. Thus, we have that there are infinitely many shifts $\zeta(s + i\tau)$ approximating a given function $f(s) \in H_0(K)$.

The above theorem is of continuous type because τ in shifts $\zeta(s+i\tau)$ can take arbitrary real values. If τ runs over a certain discrete set, then we have the discrete universality that was proposed in [7]. Denote by $\#A$ the cardinality of a set A, and suppose that N runs over the set of non-negative integers. If K and $f(s)$ are as above, then we have, for $h>0$ and $\varepsilon>0$,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K}|\zeta(s+ikh)-f(s)|<\varepsilon\right\}>0.$$

Approximations of analytic functions by more general discrete shifts were considered in [8–10].

Denote by $\gamma_1<\gamma_2<\cdots\leqslant\gamma_k\leqslant\cdots$ the positive imaginary parts of non-trivial zeros $\rho_k=\beta_k+i\gamma_k$ of the function $\zeta(s)$. Discrete universality theorems with shifts $\zeta(s+ih\gamma_k)$ were obtained in [11,12]. In [11], for this the Riemann hypothesis was used, while in [12], the weak form of the Montgomery pair correlation conjecture [13] was involved. More precisely, the estimate, for $c>0$,

$$\sum_{\substack{0<\gamma_k,\gamma_l\leqslant T\\ |\gamma_k-\gamma_l|<c/\log T}}1\ll_c T\log T,\quad T\to\infty, \tag{1}$$

was required. Analogical results for more general functions were given in [14,15].

On the other hand, all above theorems are non-effective in the sense that any concrete shift approximating a given analytic function is not known. This shortcoming leads to the idea of universality in intervals as short as possible containing τ with approximating property. The first result in this direction was obtained in [16].

Theorem 1. *Suppose that* $T^{1/3}(\log T)^{26/15}\leqslant H\leqslant T$, $K\in\mathcal{K}$ *and* $f(s)\in H_0(K)$. *Then, for every* $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{H}\text{meas}\left\{\tau\in[T,T+H]:\sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\right\}>0.$$

The aim of this paper is the universality of the function $\zeta(s)$ in short intervals with shifts $\zeta(s+ih\gamma_k)$. In this case, the estimate (1) is not sufficient. Therefore, for $N^{1/2+\varepsilon}\leqslant M\leqslant N$ with $\varepsilon>0$, we use the following hypothesis:

$$\sum_{\substack{k=N\\ |\gamma_k-\gamma_l|<c/\log N}}^{N+M}\sum_{l=N}^{N+M}1\ll_c M, \tag{2}$$

which, as estimate (1), also gives a certain information on the pair correlation of non-trivial zeros, differently from estimate (1), however, in short intervals.

Theorem 2. *Suppose that* $N^{1/2+\varepsilon}\leqslant M\leqslant N$, *and estimate* (2) *are true. Let* $K\in\mathcal{K}$ *and* $f(s)\in H_0(K)$. *Then, for every* $\varepsilon>0$ *and* $h>0$,

$$\liminf_{N\to\infty}\frac{1}{M+1}\#\left\{N\leqslant k\leqslant N+M:\sup_{s\in K}|\zeta(s+ih\gamma_k)-f(s)|<\varepsilon\right\}>0.$$

Moreover, "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many* $\varepsilon>0$.

Theorem 2 has a generalization for certain compositions $F(\zeta(s))$. Denote by $H(D)$ the space of analytic functions on the strip D endowed with the topology of uniform convergence on compacta. Moreover, let

$$S=\{g\in H(D):\text{either } g(s)\neq 0 \text{ for all } s\in D,\ \text{or } g(s)\equiv 0\},$$

and, for the operator $F : H(D) \to H(D)$ and distinct complex numbers $a_1, \dots, a_r$,

$$H_{a_1,\dots,a_r;F}(D) = \left\{g \in H(D) : g(s) \neq a_j \text{ for all } s \in D,\ j = 1, \dots, r\right\} \cup \{F(0)\}.$$

Then we have

Theorem 3. *Suppose that estimate* (2) *is true,* $N^{1/2+\varepsilon} \leqslant M \leqslant N$*, and* $F : H(D) \to H(D)$ *is a continuous operator such that* $F(S) \supset H_{a_1,\dots,a_r;F}(D)$*. For* $r = 1$*, let* $K \in \mathcal{K}$ *and* $f(s)$ *be a continuous* $\neq a_1$ *function on* K*, and analytic in the interior of* K*. For* $r \geqslant 2$*, let* K *be an arbitrary compact subset of* D*, and* $f(s) \in H_{a_1,\dots,a_r;F}(D)$*. Then, for every* $\varepsilon > 0$ *and* $h > 0$*,*

$$\liminf_{N \to \infty} \frac{1}{M+1} \# \left\{ N \leqslant k \leqslant N+M : \sup_{s \in K} |F(\zeta(s + ih\gamma_k)) - f(s)| < \varepsilon \right\} > 0.$$

Moreover "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many* $\varepsilon > 0$*.*

For example, the operators $F(g) = \sin g$ and $F(g) = \sinh g$ satisfy the hypotheses of Theorem 3 with $a_1 = -1$ and $a_2 = 1$.

The proofs of Theorems 2 and 3 use probabilistic limit theorems for measures in the space $H(D)$. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$. The main limit theorem will be proved for

$$P_{N,M,h}(A) = \frac{1}{M+1} \# \{ N \leqslant k \leqslant N+M : \zeta(s + ih\gamma_k) \in A \},\ A \in \mathcal{B}(H(D)),$$

as $N \to \infty$. We divide its proof into four sections.

2. A Limit Theorem on the Torus

Denote by γ the unit circle on the complex plane, by $\mathbb{P}$ the set of all prime numbers, and define the set

$$\Omega = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. With the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the pth component of an element $\omega \in \Omega$, $p \in \mathbb{P}$.

In this section, we will prove a limit theorem for

$$Q_{N,M,h}(A) = \frac{1}{M+1} \# \left\{ N \leqslant k \leqslant N+M : \left(p^{-ih\gamma_k} : p \in \mathbb{P}\right) \in A \right\},\ A \in \mathcal{B}(\Omega),$$

as $N \to \infty$.

Before the statement of a limit theorem for $Q_{N,M,h}$ as $N \to \infty$, we will recall some useful results that will be used in its proof. Denote by $N(T)$ the number of non-trivial zeros of $\zeta(s)$ in the region $\{s \in \mathbb{C} : 0 < t < T\}$.

Lemma 1 (von Mongoldt formula)**.** *For* $T \to \infty$*,*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

For the proof, see, for example, [17].

Denote by $N(\sigma, T)$ the number of zeros $\boldsymbol{\rho = \beta + i\gamma}$ of $\zeta(s)$ with $\beta > \sigma$ and $|\gamma| < T$.

Lemma 2. *Suppose that $H \geqslant T^{\alpha}$ with $\alpha > 27/82$. Then, for $1/2 < \sigma < 1$, uniformly in σ,*

$$N(\sigma, T+H) - N(\sigma, T) = O\left(\frac{H}{\sigma - 1/2}\right).$$

Proof of the lemma can be found in [18].

For positive $u \neq 1$, denote by $\Lambda(u)$ the von Mongoldt function if $u \in \mathbb{N} \setminus \{1\}$, and zero, otherwise.

Lemma 3. *For positive $x \neq 1$ and $T \to \infty$,*

$$\sum_{0<\gamma_k<T} x^{\rho_k} = \left(\Lambda(x) - x\Lambda\left(\frac{1}{x}\right)\right)\frac{T}{2\pi} + O\left(T^{(1/2)+\varepsilon}\right)$$

with every $\varepsilon > 0$.

Proof. The lemma is Theorem 2 of [19] with $a = 0$. □

Lemma 4. *Suppose that $N^{1/2+\varepsilon} \leqslant M \leqslant N$ with $\varepsilon > 0$. Then, for positive $x \neq 1$, as $N \to \infty$,*

$$\sum_{k=N}^{N+M} x^{\rho_k} \ll_x \frac{M}{\sqrt{\log M}}.$$

Proof. Since

$$\frac{N}{\log N} \ll \gamma_N \ll \frac{N}{\log N},$$

in view of Lemma 3,

$$\sum_{\gamma_N<\gamma\leqslant\gamma_{N+M}} x^{\rho} = \left(\Lambda(x) - x\Lambda\left(\frac{1}{x}\right)\right)\frac{\gamma_{N+M}-\gamma_N}{2\pi} + O\left(\frac{N^{1/2+\varepsilon}}{\sqrt{\log N}}\right). \tag{3}$$

An application of Lemma 1 gives

$$N + M = \sum_{\gamma\leqslant\gamma_{N+M}} 1 = \frac{\gamma_{N+M}}{2\pi}\log\frac{\gamma_{N+M}}{2\pi e} + O(\log N)$$

and

$$N = \sum_{\gamma\leqslant\gamma_N} 1 = \frac{\gamma_N}{2\pi}\log\frac{\gamma_N}{2\pi e} + O(\log N).$$

Therefore,

$$\gamma_{N+M} = \frac{2\pi(N+M)}{\log(\gamma_{N+M}/(2\pi e))} + O(1)$$

and

$$\gamma_N = \frac{2\pi N}{\log(\gamma_N/(2\pi e))} + O(1).$$

Hence,

$$\gamma_{N+M} - \gamma_N \leqslant \frac{2\pi(N+M)}{\log(\gamma_N/(2\pi e))} - \frac{2\pi N}{\log(\gamma_N/(2\pi e))} + O(1) \ll \frac{M}{\log N} + O(1) \ll \frac{M}{\log M}. \tag{4}$$

This together with Equation (3) proves the lemma. □

Now, we state the limit theorem for $Q_{N,M,h}$.

Theorem 4. *Suppose that, for any $\varepsilon > 0$, $N^{1/2+\varepsilon} \leqslant M \leqslant N$. Then $Q_{N,M,h}$ converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. Denote by $g_{N,M,h}(\underline{k})$, $\underline{k} = (k_p : k_p \in \mathbb{Z},\ p \in \mathbb{P})$, the Fourier transform of $Q_{N,M,h}$, i.e.,

$$g_{N,M,h}(\underline{k}) = \int_{\Omega} \left(\prod_{p \in \mathbb{P}}^{*} \omega^{k_p}(p) \right) \mathrm{d}Q_{N,M,h},$$

where the star "$*$" means that only a finite number of integers k_p are distinct from zero. Thus, by the definition of $Q_{N,M,h}$,

$$g_{N,M,h}(\underline{k}) = \frac{1}{M+1} \sum_{k=N}^{N+M} \exp\left\{ -ih\gamma_k \sum_{p \in \mathbb{P}}^{*} k_p \log p \right\}. \tag{5}$$

Clearly,

$$g_{N,M,h}(\underline{0}) = 1. \tag{6}$$

Now, suppose that $k \neq \underline{0}$. Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent within the field of rational numbers $\mathbb{Q}$, in that case we have

$$a \stackrel{def}{=} \sum_{p \in \mathbb{P}}^{*} k_p \log p \neq 0.$$

Thus, we will estimate the sum

$$\sum_{k=N}^{N+M} \exp\{iha\, \gamma_k\}.$$

It is easily seen that

$$\begin{aligned} \sum_{k=N}^{N+M} \left(\exp\{ha\beta_k\} - \exp\left\{\frac{1}{2}ha\right\} \right) &\ll_{h,a} \sum_{k=N}^{N+M} \left| \exp\left\{ ha\left(\beta_k - \frac{1}{2}\right) \right\} - 1 \right| \\ &\ll_{h,a} \sum_{k=N}^{N+M} \left| \beta_k - \frac{1}{2} \right| = \sum_{k=N}^{N+M}{}' \left| \beta_k - \frac{1}{2} \right| + \sum_{k=N}^{N+M}{}'' \left| \beta_k - \frac{1}{2} \right|, \end{aligned} \tag{7}$$

where $|\beta_k - 1/2| \leqslant 1/\log\log M$ in $\sum'$, and $|\beta_k - 1/2| > 1/\log\log M$ in $\sum''$. Obviously,

$$\sum_{k=N}^{N+M}{}' \left| \beta_k - \frac{1}{2} \right| \leqslant \frac{M}{\log\log M}. \tag{8}$$

Therefore, by Lemma 2 and estimate (2),

$$\sum_{k=N}^{N+M}{}'' \left| \beta_k - \frac{1}{2} \right| \ll \sum_{\gamma_N < \gamma \leqslant \gamma_{N+M}}{}'' 1 \ll \frac{M \log\log M}{\log M}.$$

This, and estimates (7) and (8) show that

$$\sum_{k=N}^{N+M} \exp\{(\beta_k + i\gamma_k)ha\} - \sum_{k=N}^{N+M} \exp\left\{ \left(\frac{1}{2} + i\gamma_k \right) ha \right\} \ll_{h,a} \frac{M}{\log\log M}. \tag{9}$$

Lemma 4 with $x = \exp\{ha\}$ implies

$$\sum_{k=N}^{N+M} \exp\{(\beta_k + i\gamma_k)ha\} \ll_{h,a} \frac{M}{\sqrt{\log M}}.$$

Therefore, in view of estimate (9),

$$\sum_{k=N}^{N+M} \exp\{iha\gamma_k\} \ll_{h,a} \sum_{k=N}^{N+M} \exp\left\{\left(\frac{1}{2}+i\gamma_k\right)ha\right\} \ll_{h,a} \frac{M}{\log\log M}.$$

Thus, by Equation (5),

$$g_{N,M,h}(\underline{k}) \ll_{h,a} \frac{1}{\log\log M}.$$

This together with Equation (6) shows that

$$\lim_{N\to\infty} g_{N,M,h}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k}=\underline{0}, \\ 0 & \text{if } \underline{k}\neq\underline{0}, \end{cases}$$

and the lemma is proved because the right-hand side of the latter equality is the Fourier transform of the measure m_H. □

3. A Limit Theorem for Absolutely Convergent Series

Let $\theta > 1/2$ be a fixed number, and $v_n(m) = \exp\{-(m/n)^\theta\}$ for $m, n \in \mathbb{N}$. Extend the function $\omega(p)$ to the set $\mathbb{N}$ by setting

$$\omega(m) = \prod_{\substack{p^l | m \\ p^{l+1}\nmid m}} \omega^l(p),$$

and define

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s}$$

and

$$\zeta_n(s,\omega) = \sum_{m=1}^{\infty} \frac{\omega(m)v_n(m)}{m^s}.$$

Then the latter series are absolutely convergent for $\sigma > 1/2$ [5]. Consider the function $u_n : \Omega \to H(D)$ defined by

$$u_n(\omega) = \zeta_n(s,\omega).$$

The absolute convergence of the series $\zeta_n(s,\omega)$ implies the continuity of u_n.

For $A \in \mathcal{B}(H(D))$, define

$$P_{N,M,n,h}(A) = \frac{1}{M+1}\#\left\{N \leqslant k \leqslant N+M : \zeta_n(s+ih\gamma_k) \in A\right\}.$$

Theorem 5. *Suppose that $N^{1/2+\varepsilon} \leqslant M \leqslant N$. Then $P_{N,M,n,h}$ converges weakly to the measure $m_H u_n^{-1} \stackrel{def}{=} V_n$.*

Proof. The theorem follows from the equality

$$P_{N,M,n,h}(A) = Q_{N,M,h}(u_n^{-1}A) = Q_{N,M,h}u_n^{-1}(A), \quad A \in \mathcal{B}(H(D)),$$

continuity of the function u_n, Theorem 4 and Theorem 5.1 of [20]. □

The weak convergence of $P_{N,M,h}$ is closely connected to that of V_n as $n \to \infty$. Define

$$\zeta(s,\omega) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

Then $\zeta(s,\omega)$ is an $H(D)$-valued random element on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ [5]. We recall that the latter infinite product, for almost all ω, is uniformly convergent on compact subsets $K \subset D$. Denote by P_ζ the distribution of the random element $\zeta(s,\omega)$, i.e.,

$$P_\zeta(A) = m_H\{\omega \in \Omega : \zeta(s,\omega) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

The following statement is very important.

Proposition 1. *The probability measure V_n converges weakly to measure P_ζ as $n \to \infty$.*

Proof. For $A \in \mathcal{B}(H(D))$, define

$$R_T(A) = \frac{1}{T}\text{meas}\{\tau \in [0,T] : \zeta(s+i\tau) \in A\}.$$

It is known that R_T, as $T \to \infty$, converges weakly to P_ζ [5]. Moreover, R_T, as $T \to \infty$, and V_n, as $n \to \infty$, converge weakly to the same probability measure on $(H(D), \mathcal{B}(H(D)))$. Thus, V_n converges weakly to P_ζ as $n \to \infty$. □

4. Mean Square Estimates in Short Intervals

To derive the weak convergence of $P_{N,M,h}$ from that of $P_{N,M,n,h}$ as $N \to \infty$, the estimate for

$$\sum_{k=N}^{N+M} |\zeta(\sigma + ih\gamma_k + it)|^2$$

with $t \in \mathbb{R}$ is needed.

We will use the following mean square estimate in short intervals.

Lemma 5. *Suppose that σ, $1/2 < \sigma < 1$, is fixed and $T^{1/3}(\log T)^{26/15} \leqslant H \leqslant T$. Then, uniformly in H,*

$$\int_{T-H}^{T+H} |\zeta(\sigma+it)|^2 \ll_\sigma H.$$

The lemma follows from Theorem 7.1 of [21], and was used in [16].

Lemma 6. *Suppose that $N^{1/2+\varepsilon} \leqslant M \leqslant N$ and estimate (2) is true. Then, for every fixed σ, $1/2 < \sigma < 1$, $h > 0$ and $t \in \mathbb{R}$,*

$$\sum_{k=N}^{N+M} |\zeta(\sigma + ih\gamma_k + it)| \ll_{\sigma,h} M(1+|t|).$$

Proof. We will apply the Gallagher lemma connecting discrete mean squares with those continuous of some functions; for the proof, see Lemma 1.4 of [22]. Let T_0, $T \geqslant \delta > 0$ be real numbers, $\mathcal{T} \neq \varnothing$ be a finite set in the interval $[T_0 + \delta/2, T_0 + T - \delta/2]$,

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x|<\delta}} 1$$

and let $S(x)$ be a complex-valued continuous function on $[T_0, T+T_0]$ having a continuous derivative on $(T_0, T+T_0)$. Then the Gallagher lemma asserts that

$$\sum_{t\in\mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leqslant \frac{1}{\delta}\int_{T_0}^{T_0+T} |S(x)|^2\,\mathrm{d}x + \left(\int_{T_0}^{T_0+T} |S(x)|^2\,\mathrm{d}x \int_{T_0}^{T_0+T} |S'(x)|^2\,\mathrm{d}x\right)^{1/2}. \quad (10)$$

We apply the Gallagher lemma for the function $\zeta(s + ikh\gamma_k + it)$. In our case $\delta = c/\log N$, $T_0 = h\gamma_N - \delta/2$, $T = h\gamma_{N+M} - h\gamma_N + \delta/2$ and $\mathcal{T} = \{h\gamma_N, h\gamma_{N+1}, \dots, h\gamma_{N+M}\}$. By estimate (2), we have

$$\sum_{k=N}^{N+M} N_\delta(h\gamma_k) = \sum_{k=N}^{N+M} \sum_{\substack{l=N \\ |\gamma_k-\gamma_l|<c/(h\log N)}}^{N+M} 1 \ll_h M. \tag{11}$$

Now, an application of the Gallagher lemma gives

$$\begin{aligned}\sum_{k=N}^{N+M} |\zeta(\sigma + ih\gamma_k + it)| &= \sum_{k=N}^{N+M} \sqrt{N_\delta(h\gamma_k)N^{-1}(h\gamma_k)}|\zeta(\sigma + ih\gamma_k + it)| \\ &\leqslant \left(\sum_{k=N}^{N+M} N_\delta(h\gamma_k) \sum_{k=N}^{N+M} N^{-1}(h\gamma_k)|\zeta(\sigma + ih\gamma_k + it)|^2\right)^{1/2} \\ &\ll_h \sqrt{M}\sqrt{\log N}\left(\int_{h\gamma_N-\delta/2}^{h\gamma_{N+M}} |\zeta(\sigma + i\tau + it)|^2 \, d\tau \right. \\ &\quad \left. + \left(\int_{h\gamma_N-\delta}^{h\gamma_{N+M}} |\zeta(\sigma + i\tau + it)|^2 d\tau \int_{h\gamma_N-\delta}^{h\gamma_{N+M}} |\zeta'(\sigma + i\tau + it)|^2 \, d\tau\right)^{1/2}\right)^{1/2}. \end{aligned} \tag{12}$$

The estimate (4) gives with certain $c_h > 0$

$$\int_{h\gamma_N-\delta}^{h\gamma_{N+M}} |\zeta(\sigma + i\tau + it)|^2 dt \ll \int_{h\gamma_N-\delta-|t|}^{h\gamma_N+c_h(M/\log M)+|t|} |\zeta(\sigma + i\tau)|^2 \, d\tau. \tag{13}$$

If $c_h(M/\log M) + |t| \leqslant h\gamma_N$, then, in view of Lemma 5, the right-hand side of (13) is

$$\ll_{\sigma,h} \frac{M}{\log M} + |t| \ll_{\sigma,h} \frac{M}{\log M}(1 + |t|).$$

If $c_h(M/\log M) + |t| > h\gamma_N$, then

$$h\gamma_N + c_h\frac{M}{\log M} + |t| < 2\left(c_h\frac{M}{\log M} + |t|\right)$$

and

$$h\gamma_N - \delta > h\gamma_N - 2c_h\frac{M}{\log M} - 2|t| > -h\gamma_N c_h\frac{M}{\log M} - |t| > -2\left(c_h\frac{M}{\log M} + |t|\right).$$

Thus, in this case,

$$\int_{h\gamma_N-\delta}^{h\gamma_{N+M}} |\zeta(\sigma + i\tau + it)|^2 \, d\tau \ll_h \int_{-2(c_h(M/\log M)+|t|)}^{2(c_h(M/\log M)+|t|)} |\zeta(\sigma + i\tau)|^2 \, d\tau \ll_{\sigma,h} \frac{M}{\log M}(1 + |t|).$$

This together with estimate (13) shows that

$$\int_{h\gamma_N-\delta}^{h\gamma_{N+M}} |\zeta(\sigma + i\tau + it)|^2 \, d\tau \ll_{\sigma,h} \frac{M}{\log M}(1 + |t|). \tag{14}$$

Estimate (14) and an application of the Cauchy integral formula lead to the bound

$$\int_{h\gamma_N-\delta}^{h\gamma_{N+M}} |\zeta'(\sigma + i\tau + it)|^2 \, d\tau \ll_{\sigma,h} \frac{M}{\log M}(1 + |t|).$$

This, estimate (14) and (12) prove the lemma. □

Now, we are ready to state an approximation lemma.

5. Approximation in the Mean

Denote by ρ the metric in $H(D)$ which induces the topology of uniform convergence on compacta. More precisely, for $g_1, g_2 \in H(D)$,

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\} \subset D$ is a sequence of compact subsets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and every compact $K \subset D$ lies in a certain K_l.

Lemma 7. *Suppose that $N^{1/2+\varepsilon} \leqslant M \leqslant N$ and (2) is true. Then, for every $h > 0$,*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{M+1} \sum_{k=N}^{N+M} \rho\left(\zeta(s + ih\gamma_k), \zeta_n(s + ih\gamma_k)\right) = 0.$$

Proof. In view of the definition of the metric ρ, it suffices to show that, for every compact $K \subset D$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{M+1} \sum_{k=N}^{N+M} \sup_{s\in K} |\zeta(s + ih\gamma_k) - \zeta_n(s + ih\gamma_k)| = 0. \tag{15}$$

Thus, let $K \subset D$ be a fixed compact set. Denote the points of K by $s = \sigma + iv$, and fix $\varepsilon > 0$ such that $\mathbf{1/2 + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon}$ for $s \in K$. It is known [5] that

$$\zeta_n(s) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s+z) l_n(z) \frac{\mathrm{d}z}{z},$$

where

$$l_n(s) = \frac{s}{\theta} \Gamma(s/\theta) n^s,$$

$\Gamma(s)$ is the Euler gamma-function, and θ comes from the definition of $v_n(m)$. Let $\theta_1 > 0$. From this, we have

$$\zeta(s) - \zeta_n(s) = \frac{1}{2\pi i} \int_{-\theta - i\infty}^{-\theta + i\infty} \zeta(s+z) l_n(z) \frac{\mathrm{d}z}{z} + R_n(s),$$

with

$$R_n(s) = \frac{l_n(1-s)}{1-s}.$$

Therefore, as in the proof of Lemma 12 of [16], we find that

$$\begin{aligned}
&\frac{1}{M+1} \sum_{k=N}^{N+M} \sup_{s\in K} |\zeta(s + ih\gamma_k) - \zeta_n(s + ih\gamma_k)| \\
&\quad \ll \int_{-\infty}^{\infty} \frac{1}{M} \sum_{k=N}^{N+M} \left|\zeta\left(\frac{1}{2} + \varepsilon + i(h\gamma_k + t)\right)\right| \sup_{s\in K} \frac{|l_n(1/2 + \varepsilon - s + it)|}{|1/2 + \varepsilon - s + it|} \,\mathrm{d}t \\
&\quad + \frac{1}{M} \sum_{k=N}^{N+M} \sup_{s\in K} |R_n(s + ih\gamma_k)| \stackrel{def}{=} I_1 + I_2.
\end{aligned} \tag{16}$$

Denote by $c_1, c_2, \ldots$ positive constants. In view of the well-known estimate

$$\Gamma(\sigma + it) \ll \exp\{-c_1|t|\}, \tag{17}$$

we find that

$$\frac{|l_n(1/2+\varepsilon-s+it)|}{|1/2+\varepsilon-s+it|} \ll n^{-\varepsilon}\exp\{-c_2|t-v|\} \ll_{K,\varepsilon} n^{-\varepsilon}\exp\{-c_3|t|\}.$$

Therefore, by Lemma 5,

$$I_1 \ll_{K,\varepsilon} n^{-\varepsilon}\int_{-\infty}^{\infty}(1+|t|)\exp\{-c_3|t|\}\,\mathrm{d}t \ll_{K,\varepsilon} n^{-\varepsilon}. \tag{18}$$

Similarly, taking into account inequality (17), we find

$$\begin{aligned} I_2 &\ll \frac{n^{1/2-2\varepsilon}}{M}\sum_{k=N}^{N+M}\exp\{-c_4|h\gamma_k-v|\} \ll_K \frac{n^{1/2-2\varepsilon}}{M}\sum_{k=N}^{N+M}\exp\{-c_5h\gamma_k\} \\ &\ll_K \frac{n^{1/2-2\varepsilon}}{M}\sum_{k=N}^{N+M}\exp\{-c_6h(k/\log k)\} \ll_{K,h} \frac{n^{1/2-2\varepsilon}}{M}. \end{aligned}$$

This, Equations (18) and (16) prove (15). □

6. A Limit Theorem for $\zeta(s)$

Using the results of Sections 3 and 4 leads to a limit theorem for $P_{N,M,h}$.

Theorem 6. *Suppose that $N^{1/2+\varepsilon} \leqslant M \leqslant N$ and estimate (2) is true. Then $P_{N,M,h}$ converges weakly to P_ζ as $N \to \infty$.*

Proof. In a certain probability space with measure μ define the random variable $\theta_{N,M,h}$ with the distribution

$$\mu\{\theta_{N,M,h} = h\gamma_k\} = \frac{1}{M+1}, \quad k = N, N+1, \ldots, N+M,$$

and consider the $H(D)$-valued random element

$$X_{N,M,n,h} = X_{N,M,n,h}(s) = \zeta_n(s+i\theta_{N,M,h}).$$

Moreover, let $X_n = X_n(s)$ be the $H(D)$-valued random element with the distribution V_n. Then, by Theorem 5,

$$X_{N,M,n,h} \xrightarrow[N\to\infty]{\mathcal{D}} X_n, \tag{19}$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution. Moreover, by Proposition 1,

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P_\zeta. \tag{20}$$

Define one more $H(D)$-valued random element

$$X_{N,M,h} = X_{N,M,h}(s) = \zeta(s+i\theta_{N,M,h}).$$

Then, using Lemma 7, we find that, for every $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty}\limsup_{N\to\infty}\mu\left\{\rho(X_{N,M,h}, X_{N,M,n,h}) \geqslant \varepsilon\right\} \\ &\leqslant \lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{\varepsilon(M+1)}\sum_{k=N}^{N+M}\rho(\zeta(s+ih\gamma_k), \zeta_n(s+ih\gamma_k)) = 0. \end{aligned}$$

Now, this, Equations (19) and (20) together with Theorem 4.2 of [20] show that

$$X_{N,M,h} \xrightarrow[n\to\infty]{\mathcal{D}} P_\zeta,$$

and theorem is proved. □

For $A \in \mathcal{B}(H(D))$, define

$$P_{N,M,h,F}(A) = \frac{1}{M+1}\#\left\{N \leqslant k \leqslant N+M : F(\zeta(s+ih\gamma_k)) \in A\right\}.$$

Corollary 1. *Suppose that $F : H(D) \to H(D)$ is a continuous operator, and (2) is true. Then $P_{N,M,h,F}$ converges weakly to $P_\zeta F^{-1}$ as $N \to \infty$.*

Proof. The corollary follows from Theorem 5, continuity of F, equality

$$P_{N,M,h,F} = P_{N,M,h}F^{-1},$$

and Theorem 5.1 of [20]. □

7. Proof of Universality

Theorems 2 and 3 are derived from Theorem 6 and Corollary 1, respectively, by using the Mergelyan theorem on the approximation of analytic functions by polynomials [23].

Proof of Theorem 2. We recall that

$$S = \{g \in H(D) : \text{either } g(s) \neq 0 \text{ for all } s \in D, \text{ or } g(s) \equiv 0\},$$

It is well known, see, for example, [5], that the support of the measure P_ζ is the set S. Define the set

$$G_\varepsilon = \left\{g \in H(D) : \sup_{s\in K}\left|g(s) - \mathrm{e}^{p(s)}\right| < \frac{\varepsilon}{2}\right\},$$

where $p(s)$ is a polynomial. Obviously, $\mathrm{e}^{p(s)} \in S$. Therefore, G_ε is an open neighbourhood of an element of the support of the measure P_ζ. Thus, by a property of the support,

$$P_\zeta(G_\varepsilon) > 0. \tag{21}$$

This, Theorem 6 and the equivalent of weak convergence in terms of open sets show that

$$\liminf_{N\to\infty} P_{N,M,h}(G_\varepsilon) \geqslant P_\zeta(G_\varepsilon) > 0.$$

Hence, by the definition of $P_{N,M,h}$ and G_ε,

$$\liminf_{N\to\infty}\frac{1}{M+1}\#\left\{N \leqslant k \leqslant N+M : \sup_{s\in K}\left|\zeta(s+ih\gamma_k) - \mathrm{e}^{p(s)}\right| < \frac{\varepsilon}{2}\right\} > 0. \tag{22}$$

Now, we apply the Mergelyan theorem and choose the polynomial $p(s)$ satisfying

$$\sup_{s\in K}\left|f(s) - \mathrm{e}^{p(s)}\right| < \frac{\varepsilon}{2}. \tag{23}$$

This and inequality (22) prove the first part of the theorem.

To prove the second part of the theorem, define the set

$$\hat{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then the set $\hat{G}_\varepsilon$ is a continuity set of the measure P_ζ for all but at most countably many $\varepsilon > 0$. This remark, Theorem 6 and the equivalent of weak convergence of probability measures in terms of open sets show that

$$\lim_{N \to \infty} P_{N,M,h}(\hat{G}_\varepsilon) = P_\zeta(\hat{G}_\varepsilon) \tag{24}$$

for all but at most countably many $\varepsilon > 0$. Inequality (23) implies the inclusion $G_\varepsilon \subset \hat{G}_\varepsilon$. Therefore, in view of inequality (21), we have $P_\zeta(\hat{G}_\varepsilon) > 0$. This, Equation (24) and the definitions of $P_{N,M,h}$ and $\hat{G}_\varepsilon$ prove the second part of the theorem. □

Proof of Theorem 3. Denote by S_F the support of the measure $P_\zeta F^{-1}$. We observe that S_F contains the closure of the set $H_{a_1,\dots,a_r;F}(D)$. Actually, let $g \in H_{a_1,\dots,a_r;F}(D)$ and G be any open neighborhood of g. Then the set $F^{-1}G$ is open as well, and lies in S. Hence, $P_\zeta(F^{-1}G) > 0$ because S is the support of P_ζ. Therefore,

$$P_\zeta F^{-1}(G) = P_\zeta(F^{-1}G) > 0.$$

This shows that S_F contains the set $H_{a_1,\dots,a_r;F}(D)$ and its closure.

Case $r = 1$. By the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{25}$$

Then, $p(s) \neq a_1$ for all $s \in K$ if ε is small enough. Therefore, by the Mergelyan theorem again, we find a polynomial $q(s)$ such that

$$\sup_{s \in K} \left| (p(s) - a_1) - \mathrm{e}^{q(s)} \right| < \frac{\varepsilon}{4}. \tag{26}$$

Since $g_1(s) \stackrel{def}{=} \mathrm{e}^{q(s)} + a_1 \in H_{a_1;F}(D)$, the set

$$\mathcal{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - g_1(s)| < \frac{\varepsilon}{2} \right\}$$

is an open subset of S_F. Hence,

$$P_\zeta F^{-1}(\mathcal{G}_\varepsilon) > 0. \tag{27}$$

This inequality together with Corollary 1, inequalities (25) and (26) prove the theorem in the case of the lower density.

In the case of density, consider the set $\hat{G}_\varepsilon$ defined in the proof of Theorem 2 which is a continuity set of the measure $P_\zeta F^{-1}$ for all but at most countably many $\varepsilon > 0$. Therefore, by Corollary 1,

$$\lim_{N \to \infty} P_{N,M,h,F}(\hat{G}_\varepsilon) = P_\zeta F^{-1}(\hat{G}_\varepsilon). \tag{28}$$

Inequalities (25) and (26) show that $\mathcal{G}_\varepsilon \subset \hat{G}_\varepsilon$. Thus, by inequality (27), $P_\zeta F^{-1}(\hat{G}_\varepsilon) > 0$. This, Equation (28) and the definitions of $P_{N,M,h,F}$ and $\hat{G}_\varepsilon$ prove the theorem in the case of density.

Case $r \geqslant 2$. In this case, the function $f(s)$ lies in S_F. Therefore, the Mergelyan theorem is not needed, and the theorem follows immediately from Corollary 1. □

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