Nonlinear Analysis: Modelling and Control, Vol. 25, No. 6, 1059–1078
https://doi.org/10.15388/namc.2020.25.20565

ISSN: 1392-5113
eISSN: 2335-8963

Estimation of the Hurst index of the solutions of fractional SDE with locally Lipschitz drift

Kęstutis Kubilius

Faculty of Mathematics and Informatics, Vilnius University,
Akademijos 4, LT-08663 Vilnius, Lithuania
kestutis.kubilius@mif.vu.lt

Received: February 9, 2020 / **Revised:** July 28, 2020 / **Published online:** November 1, 2020

Abstract. Strongly consistent and asymptotically normal estimate of the Hurst index H are obtained for stochastic differential equations (SDEs) that have a unique positive solution. A strongly convergent approximation of the considered SDE solution is constructed using the backward Euler scheme. Moreover, it is proved that the Hurst estimator preserves its properties, if we replace the solution with its approximation.

Keywords: fractional Brownian motion, Hurst index, backward Euler approximation, fractional Ait-Sahalia model, fractional CKLS model.

1 Introduction

The models defined by SDE

$$X_t = x_0 + \int_0^t g(X_s)\,\mathrm{d}s + \sigma \int_0^t X_s^\beta\,\mathrm{d}B_s, \quad \frac{1}{2} \leqslant \beta < 1,$$

where B is standard Brownian motion, g is continuous function on $(0, \infty)$, $x_0 > 0$ is nonrandom initial value, $\sigma > 0$ is a constant, include several well-known models such as Chan–Karolyi–Longstaff–Sanders (CKLS), Cox–Ingersoll–Ross (CIR), Ait-Sahalia and others, which are widely used in many financial applications. These models were studied in [1, 2, 13–15] and in the references therein. They also found conditions when the solutions to these models are positive. In addition, implicit numerical schemes preserving positivity were also considered. To solve these problems, some authors convert the original SDE using the Lamperti transform to SDE with a constant diffusion coefficient. This approach is also convenient for considering fractional analogues of the CKLS, CIR, Ait-Sahalia models.

Currently, much attention is paid to models with fractional Brownian motion (fBm) B^H since it introduces a memory element into the model under consideration. Consider SDE

$$X_t = x_0 + \int_0^t g(X_s)\,\mathrm{d}s + \sigma \int_0^t X_s^\beta\,\mathrm{d}B_s^H, \quad \frac{1}{2} \leqslant \beta < 1, \tag{1}$$

with $H \in (1/2, 1)$. The stochastic integral in equation (1) is a pathwise Riemann–Stieltjes integral. SDE (1) cannot be treated directly since the function $h(x) = x^\beta$, $1/2 \leqslant \beta < 1$, does not satisfy the usual Lipschitz conditions that are commonly imposed.

For fractional CIR and CKLS models, the existence of a unique positive solution of equation (1) was obtained in [3, 8, 9, 11, 12, 16]. The proof is based on several approaches. One approach is based on the consideration of the conditions under which the equation

$$Y_t = y_0 + \int_0^t f(s, Y_s)\,\mathrm{d}s + B_t^H, \quad H \in \left(\frac{1}{2}, 1\right),$$

admits a unique positive solution, where $f(t, x)$ is a locally Lipschitz function with respect to the space variable x on $x \in (0, \infty)$. This approach was used in [3, 8, 16], where the inverse Lamperti transform was used to obtain conditions under which equation (1) admits a unique positive solution for fractional CIR and CKLS models. Unfortunately, we cannot apply the proof of the positivity of the solution of equation (1) given in [16]. The proof must be revised because it is not applicable, for example, for the Ait-Sahalia model with $1/2 \leqslant \beta < 1$.

Marie [9] used rough-path approach to find the existence of the unique positive solution of the fractional CKLS model. One more approach for fractional CIR process was suggested in [11, 12], where the integral with respect to fractional Brownian motion is considered as the pathwise Stratonovich integral. In [10], it was proved that equation

$$X_t = x_0 + b \int_0^t X_s\,\mathrm{d}s + \sigma \int_0^t X_s^\beta\,\mathrm{d}B_s^H, \quad x_0 > 0,$$

has a unique solution for $H \in (1/(1+\beta),\ 1)$, $\beta \in [1/2, 1)$, and $X_t = 0$ a.s. for all $t \geqslant \tau$, where $\tau = \inf\{t > 0\colon\ X_t = 0\}$.

The problem of the statistical estimation of the long-memory parameter H is of a great importance. This parameter determines the mathematical properties of the model and consequently describes the behavior of the underlying physical system.

Our goal is to construct strongly consistent and asymptotically normal estimator of the Hurst index H for SDE (1), which has a unique positive solution. For such processes, we can do this in the same way as done for the diffusion coefficient satisfying the usual Lipschitz conditions (see [6, 7]). More results on parameter estimations for stochastic differential equations can be found in the book [5]. Since the existence of a unique positive solution for general form SDE (1) is unknown, we will pay attention to this problem.

To model the estimator of the Hurst index, we need an approximation of the SDE (1) solution. The approximation of the solution X is based on the use of the backward Euler scheme, which is positivity preserving. Moreover, the Hurst index H estimator preserves its properties if we replace the solution X with its approximation.

The paper is organized in the following way. In Section 2, we present the main results of the paper. In Section 3, we prove the main auxiliary result about the existence and uniqueness of positive solution for SDE (1). Section 4 contains proofs of main theorems. In Section 5, fractional CKL and Ait-Sahalia models are considered as examples. Section 6 gives some examples of simulating the fractional CKLS model to illustrate the results. Finally, in Appendix, we recall same results for fBm and the Love–Young inequality.

2 Main results

To state our main results, we use the following requirements on function f:

(C1) A function $f(x)$ is a continuously differentiable on $(0, +\infty)$.

(C2) There exist constants $a > 0$ and $\alpha \geqslant 0$ such that $f(x) \geqslant a/x^{1+\alpha}$ for all sufficiently small x.

(C3) There exists a constant $K \in \mathbb{R}$ such that the derivative is bounded above by K, i.e., $f'(x) \leqslant K$.

To estimate the Hurst index for SDE (1), we need conditions when this equation admits a unique positive solution. The following theorem solves this problem. $\mathcal{C}^\gamma([0, T])$ denotes the space of Hölder continuous functions of order $\gamma > 0$ on $[0, T]$.

Theorem 1. *Assume that function f satisfies conditions* (C1)–(C3). *Then the equation*

$$X_t = x_0 + \int_0^t g(X_s)\,\mathrm{d}s + \sigma \int_0^t X_s^\beta \,\mathrm{d}B_s^H, \quad t \geqslant 0, \tag{2}$$

where $g(x) = x^\beta f(x^{1-\beta})$, $1/2 \leqslant \beta < 1$, is well defined and has a unique positive solution $X \in \mathcal{C}^\gamma([0, T])$ with order $\gamma \in (1/2, H)$, $H \in (1/2, 1)$.

Our goal is to construct strongly consistent and asymptotically normal estimator of the Hurst parameter H for the solution X of equation (2) from discrete observations of a single sample path.

Let $\pi = \{t_k^n = (k/n)T,\ 1 \leqslant k \leqslant n\}$ be a sequence of uniform partitions of the interval $[0, T]$ and $h = t_k^n - t_{k-1}^n$, $1 \leqslant k \leqslant n$. For a real-valued process $X = \{X_t, t \in [0, T]\}$, we define the first and second-order increments along uniform partitions as

$$\Delta_{n,k} X = X_{t_k^n} - X_{t_{k-1}^n}, \quad 1 \leqslant k \leqslant n,$$
$$\Delta_{n,k}^{(2)} X = X_{t_{k+1}^n} - 2X_{t_k^n} + X_{t_{k-1}^n}, \quad 1 \leqslant k \leqslant n-1.$$

To avoid cumbersome expressions, we introduce symbol O_ω. Let (Z_n) be a sequence of r.v., ς is an a.s. nonnegative r.v. and $(a_n) \subset (0,\infty)$ vanishes. $Z_n = O_\omega(a_n)$ means that $|Z_n| \leqslant \varsigma \cdot a_n$. In particular, $Z_n = O_\omega(1)$ corresponds to the sequence (Z_n), which is a.s. bounded.

Theorem 2. *Assume that X is a unique positive solution of SDE* (2) *with $H \in (1/2, 1)$. Then*

$$\widehat{H}_n = H + O_\omega\biggl(\biggl(\frac{\ln n}{n}\biggr)^{1/2}\biggr), \qquad 2\ln 2\sqrt{n}\,(\widehat{H}_n - H) \xrightarrow{\mathrm{d}} \mathcal{N}(0, \sigma_H^2)$$

with a known variance σ_H^2 defined in Section A.2, *where*

$$\widehat{H}_n = \frac{1}{2} - \frac{1}{2\ln 2}\ln\frac{\widetilde{V}^{(2)X}_{2n,T}}{\widetilde{V}^{(2)X}_{n,T}}, \qquad \widetilde{V}^{(2)X}_{n,T} = \sum_{k=1}^{n-1}\biggl(\frac{\Delta^{(2)}_{n,k}X}{X^\beta(t_k^n)}\biggr)^2.$$

In practice, it is very interesting to compare different estimators. Therefore, we consider approximations of discrete time that can be used in modeling. To construct an approximation scheme for the SDE (2) solution, we use the solution of the SDE

$$Y_t = y_0 + (1-\beta)\int_0^t f(Y_s)\,\mathrm{d}s + (1-\beta)\sigma B_t^H, \quad t \geqslant 0,\ H \in \biggl(\frac{1}{2}, 1\biggr), \qquad (3)$$
$$y_0 = x_0^{1-\beta}.$$

The solutions of SDEs (2) and (3) satisfy the relation $X_t = Y_t^{1/(1-\beta)}$ (see the proof of Theorem 1).

The backward Euler approximation scheme for Y is defined as follows:

$$\widehat{Y}_{n,k+1} = \widehat{Y}_{n,k} + (1-\beta)f(\widehat{Y}_{n,k+1})h + \sigma(1-\beta)\Delta_{n,k+1}B^H, \quad 0 \leqslant k \leqslant n-1,$$
$$\widehat{Y}_{n,0} = y_0.$$

For the well-definedness of the backward Euler approximation scheme, we need the following assumption:

(C4) Set $F(x) = x - (1-\beta)f(x)h$ on $(0,\infty)$. Assume that the function $f(x)$ satisfies conditions (C1), (C3) and there exists $h_0 > 0$ such that $\lim_{x\to+\infty} F(x) = +\infty$, $\lim_{x\to 0^+} F(x) = -\infty$ for $0 < h < h_0$.

Remark 1. Note that under condition (C3) the function $F(x)$ is strictly monotone on $(0,\infty)$ for small h. Thus, from the conditions (C3) and (C4) it follows that for each $b \in \mathbb{R}$, the equation $F(x) = b$ has a unique positive solution for $0 < h < h_0$. Consequently, the backward Euler approximation scheme preserves positivity.

Theorem 3. *Let the function $f(x)$ satisfies conditions* (C1)–(C4)*, and f' is continuous on $(0,\infty)$. If the sequence of uniform partitions π of the interval $[0,T]$ is such that $h<h_0$, then for any $T>0$,*

$$\sup_{0\leqslant t\leqslant T}\bigl|Y_t-\widehat{Y}_t^n\bigr| = O_\omega\bigl(n^{-\gamma}\bigr),\quad \gamma\in\left(\frac{1}{2},H\right), \tag{4}$$

where

$$\widehat{Y}_t^n = \widehat{Y}_{n,k} + \frac{t-t_k^n}{h}(\widehat{Y}_{n,k+1}-\widehat{Y}_{n,k}),\quad t\in\bigl(t_k^n,t_{k+1}^n\bigr],\ k=0,\dots,n-1,$$
$$\widehat{Y}_{n,0}^n = y_0.$$

Moreover,

$$\sup_{0\leqslant t\leqslant T}\bigl|X_t-\bigl(\widehat{Y}_t^n\bigr)^{1/(1-\beta)}\bigr| = O_\omega\bigl(n^{-\gamma}\bigr),\quad \gamma\in\left(\frac{1}{2},H\right), \tag{5}$$

where X is the solution of equation (2)*.*

The following result states that if we replace the solution X with its approximation, then the estimator of the Hurst index H will remain strongly consistent and asymptotically normal.

Theorem 4. *Let conditions of Theorem* 3 *are satisfied. Moreover, let $\beta=(m-1)/m$, $m\in\mathbb{N}$. Then*

$$\widehat{H}_n^E = H + O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right),\qquad 2\ln 2\sqrt{n}\bigl(\widehat{H}_n^E-H\bigr)\xrightarrow{\mathrm{d}}\mathcal{N}\bigl(0,\sigma_H^2\bigr)$$

with a known variance σ_H^2 defined in Section A.2*, where*

$$\widehat{H}_n^E = \frac{1}{2}-\frac{1}{2\ln 2}\ln\frac{\widetilde{V}_{2n,T}^{E(2)\widehat{Y}}}{\widetilde{V}_{n,T}^{E(2)\widehat{Y}}},\qquad \widetilde{V}_{n,T}^{E(2)\widehat{Y}} = \sum_{k=1}^{n-1}\left(\frac{\Delta_{n,k}^{(2)}\widehat{Y}^m}{\widehat{Y}_k^{m-1}}\right)^2,$$

$\Delta_{n,i}^{(2)}\widehat{Y}^m = \widehat{Y}_{n,i+1}^m - 2\widehat{Y}_{n,i}^m + \widehat{Y}_{n,i-1}^m$, $(\widehat{Y}_{n,i})_{0\leqslant i\leqslant n}$ *is the backward Euler approximation of the process Y.*

3 Auxiliary result

We are interested in conditions under which the SDE

$$Y_t = y_0 + k_1\int_0^t f(Y_s)\,\mathrm{d}s + k_2 B_t^H,\quad H\in\left(\frac{1}{2},1\right), \tag{6}$$

has a unique positive solution, where k_1 and k_2 are positive constants. As mentioned in the introduction, this type of equation was considered in [3, 8, 16]. We provide conditions of a different kind than in the above papers under which equation (6) has a unique positive solution and which are easily applicable to the fractional CKLS and Ait-Sahalia models.

Consider the following deterministic differential equation driven by a continuous function φ:

$$y_t = y_0 + k_1 \int_0^t f(y_s)\,\mathrm{d}s + k_2\varphi_t, \tag{7}$$

where y_0 is a constant.

Proposition 1. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a function such that $\varphi_0 = 0$, and $\varphi \in \mathcal{C}^\lambda([0,T])$ for all $T > 0$ and fixed $\lambda \in (1/2, 1)$. Let function $f(x)$ satisfies conditions* (C1)–(C3). *If $y_0 > 0$ then equation* (7) *has a unique positive solution $y \in \mathcal{C}^\lambda([0,T])$ for any $T > 0$.*

Proof. Existence. Since $f(x)$ is locally Lipschitz continuous in $(0, +\infty)$, it is easy to see that there exists a continuous local solution y to (7) on some interval $[0, \tau)$, where $\tau = \inf\{t > 0\colon\ y_t = 0\}$. For the existence of a positive solution, we need to prove that $\tau = +\infty$. The proof is by contradiction. During the proof, we repeat the outlines of the proof of [3] (see also [2], [12]).

Under condition (C2), there exists $\varepsilon_0 > 0$ such that

$$f(x) \geqslant \frac{a}{x^{1+\alpha}} \quad \forall x \in (0, \varepsilon_0). \tag{8}$$

For any $\varepsilon \in (0, \varepsilon_0)$, let us introduce the last moment of hitting the level of ε before the first zero reaching $\tau_\varepsilon = \sup\{t \in (0,\tau)\colon\ y_t = \varepsilon\}$. Then $y_t \in (0, \varepsilon)$ for all $t \in (\tau_\varepsilon, \tau)$. This, together with (8), implies

$$f(y_t) \geqslant \frac{a}{y_t^{1+\alpha}} \quad \forall t \in (\tau_\varepsilon, \tau).$$

According to the definitions of τ, τ_ε and y, the following equality is true:

$$\varepsilon + k_1 \int_{\tau_\varepsilon}^{\tau} f(y_s)\,\mathrm{d}s = k_2(\varphi_{\tau_\varepsilon} - \varphi_\tau). \tag{9}$$

Since the function $y_t \in (0, \varepsilon)$ on the interval (τ_ε, τ), then for all $t \in (\tau_\varepsilon, \tau)$,

$$\frac{a}{y_t^{1+\alpha}} \geqslant \frac{a}{\varepsilon^{1+\alpha}}.$$

From (9) and inequality $|\varphi_t - \varphi_s| \leqslant K_\varphi |t - s|^\lambda$ it follows that

$$\varepsilon + \frac{k_1 a}{\varepsilon^{1+\alpha}}(\tau - \tau_\varepsilon) \leqslant K_\varphi k_2 (\tau - \tau_\varepsilon)^\lambda.$$

Now we show that there exists $\varepsilon^* \in (0, \varepsilon)$ such that, for all $\varepsilon < \varepsilon^*$ and for all $x \geqslant 0$, $F_\varepsilon(x) > 0$, where

$$F_\varepsilon(x) = C_{\varepsilon,\alpha} x - \widehat{C} x^\lambda + \varepsilon, \quad C_{\varepsilon,\alpha} = \frac{k_1 a}{\varepsilon^{1+\alpha}}, \quad \widehat{C} = K_\varphi k_2.$$

Then we get contradiction, which proves that the solution of equation (7) is positive.

It is easy to check that $F_\varepsilon(0) = \varepsilon > 0$ and F_ε is convex on $(0, +\infty)$ (its second derivative is strictly positive on this set), so it is enough to examine the sign of the function in its critical points. So

$$F'_\varepsilon(\tilde{x}) = C_{\varepsilon,\alpha} - \lambda\widehat{C}\tilde{x}^{\lambda-1} = 0$$
$$\implies \quad \tilde{x} = \left(\frac{C_{\varepsilon,\alpha}}{\lambda\widehat{C}}\right)^{1/(\lambda-1)} = \left(\frac{k_1 a}{K_\varphi \lambda k_2}\right)^{1/(\lambda-1)} \varepsilon^{(1+\alpha)/(1-\lambda)}.$$

After some calculations, we get

$$F_\varepsilon(\tilde{x}) = (\lambda - 1)K_\varphi k_2 \left(\frac{k_1 a}{K_\varphi \lambda k_2}\right)^{\lambda/(\lambda-1)} \varepsilon^{(1+\alpha)\lambda/(1-\lambda)} + \varepsilon.$$

Since $\lambda > 1/2$, then $\lambda/(1-\lambda) > 1$ and $(1+\alpha)\lambda/(1-\lambda) > 1$. For any $K > 0$, there exists $\varepsilon^* > 0$ such that $\varepsilon - K\varepsilon^{(1+\alpha)\lambda/(1-\lambda)} > 0$ for all $\varepsilon < \varepsilon^*$. Choosing the corresponding ε^* for

$$K := (1 - \lambda)K_\varphi k_2 \left(\frac{k_1 a}{K_\varphi \lambda k_2}\right)^{\lambda/(\lambda-1)}$$

and choosing an arbitrary $\varepsilon < \varepsilon^*$, we obtain that $F_\varepsilon(x) > 0$ for all $x > 0$. The contradiction obtained proves that $\tau = +\infty$.

It remains to prove that $y \in \mathcal{C}^\lambda([0, T])$. Indeed,

$$\begin{aligned}
|y_t - y_s| &\leqslant k_1 \int_s^t \big|f(y_u)\big| \, \mathrm{d}u + k_2\sigma|\varphi_t - \varphi_s| \\
&\leqslant k_1 \max_{0 \leqslant u \leqslant T} \big|f(y_u)\big|(t - s) + k_2\sigma C(t - s)^\lambda \\
&\leqslant \widetilde{C}(t - s)^\lambda.
\end{aligned}$$

Uniqueness. Let $\widetilde{y}$ and $\widehat{y}$ be two positive solutions of equation (7). Then

$$\widetilde{y}_t - \widehat{y}_t = \int_0^t \big(f(\widetilde{y}_s) - f(\widehat{y}_s)\big) \, \mathrm{d}s$$

and from condition (C3)

$$\begin{aligned}
(\widetilde{y}_t - \widehat{y}_t)^2 &= 2\int_0^t (\widetilde{y}_t - \widehat{y}_t)\big(f(\widetilde{y}_s) - f(\widehat{y}_s)\big) \, \mathrm{d}s \\
&\leqslant 2K \int_0^t (\widetilde{y}_t - \widehat{y}_t)^2 \, \mathrm{d}s.
\end{aligned}$$

From Gronwall 's inequality it follows that $\widetilde{y}_t = \widehat{y}_t$ for all $t \leqslant T$, $T > 0$. □

As an immediate consequence of Proposition 1, we have the following result.

Proposition 2. *Assume that conditions of Proposition* 1 *are satisfied and* $y_0 > 0$. *Then there exists a unique positive solution of equation* (6) *such that* $Y \in \mathcal{C}^\gamma([0,T])$, $T > 0$, *where* $\gamma \in (1/2, H)$, $H \in (1/2, 1)$.

4 Proofs of main theorems

Proof of Theorem 1. Set $X_t = Y_t^{1/(1-\beta)}$ and $x_0 = y_0^{1/(1-\beta)}$, where Y is a solution of equation (6) with $k_1 = k_2 = 1-\beta$. Since the process Y is positive Hölder continuous process up to the order $\gamma \in (1/2, H)$ and $\beta/(1-\beta) \geqslant 1$, then X_s^β is Hölder continuous process up to the order $\gamma \in (\frac{1}{2}, H)$. This follows from the inequality

$$\bigl|Y_t^{\beta/(1-\beta)} - Y_s^{\beta/(1-\beta)}\bigr| \leqslant \frac{\beta}{1-\beta} \sup_{0 \leqslant t \leqslant T} Y_t^{(2\beta-1)/(1-\beta)} |Y_t - Y_s| \tag{10}$$

for $1/2 \leqslant \beta < 1$, where we applied the mean value theorem.

Thus, the integral $\int_0^t X_s^\beta \, \mathrm{d}B_s^H$ is well defined as a pathwise Riemann–Stieltjes integral for $\gamma \in (1/2, H)$ and equation (2) is well defined. Now we verify that X is a solution of equation (2). By chain rule we obtain

$$\begin{aligned} X_t = Y_t^{1/(1-\beta)} &= Y_0^{1/(1-\beta)} + \frac{1}{1-\beta} \int_0^t Y_s^{\beta/(1-\beta)} \, \mathrm{d}Y_s \\ &= y_0^{1/(1-\beta)} + \int_0^t Y_s^{\beta/(1-\beta)} f(Y_s) \, \mathrm{d}s + \sigma \int_0^t Y_s^{\beta/(1-\beta)} \, \mathrm{d}B_s^H \\ &= x_0 + \int_0^t g(X_s) \, \mathrm{d}s + \sigma \int_0^t X_s^\beta \, \mathrm{d}B_s^H, \end{aligned}$$

where $g(x) = x^\beta f(x^{1-\beta})$. Thus, equation (2) has a continuous positive solution. Since

$$|X_t - X_s| = \bigl|Y_t^{1/(1-\beta)} - Y_s^{1/(1-\beta)}\bigr| \leqslant \frac{1}{1-\beta} \sup_{0 \leqslant t \leqslant T} Y_t^{\beta/(1-\beta)} |Y_t - Y_s|$$

for $1/2 \leqslant \beta < 1$, then $X \in \mathcal{C}^\gamma([0,T])$, $\gamma \in (1/2, H)$. □

The proof of Theorem 2 is based on the following lemma.

Lemma 1. *Assume that conditions of Theorem* 1 *are satisfied and* $1/2 \leqslant \beta < 1$. *Then*

$$\Delta_{n,i}^{(2)} X = \sigma X^\beta(t_i^n) \Delta_{n,i}^{(2)} B^H + O_\omega\bigl(n^{-2\gamma}\bigr),$$

where $\gamma \in (1/2, H)$.

Proof. Second-order increments of the process X we write as follows:

$$\begin{aligned}\Delta_{n,k}^{(2)} X = &\left(\int_{t_k^n}^{t_{k+1}^n} \left[g(X_s) - g(X_{t_k^n})\right] \mathrm{d}s - \int_{t_{k-1}^n}^{t_k^n} \left[g(X_s) - g(X_{t_k^n})\right] \mathrm{d}s\right) \\ &+ \sigma\left(\int_{t_k^n}^{t_{k+1}^n} \left[X_s^\beta - X_{t_k^n}^\beta\right] \mathrm{d}B_s^H - \int_{t_{k-1}^n}^{t_k^n} \left[X_s^\beta - X_{t_k^n}^\beta\right] \mathrm{d}B_s^H\right) \\ &+ \sigma X_{t_k^n}^\beta \Delta_{n,i}^{(2)} B^H.\end{aligned}$$

Applying inequality (10), condition (C1) and the fact that $Y \in \mathcal{C}^\gamma([0,T])$, $\gamma \in (1/2, H)$, we obtain

$$\begin{aligned}\left|g(X_t) - g(X_s)\right| &= \left|Y_t^{\beta/(1-\beta)} f(Y_t) - Y_s^{\beta/(1-\beta)} f(Y_s)\right| \\ &\leqslant \left|f(Y_t)\right|\left|Y_t^{\beta/(1-\beta)} - Y_s^{\beta/(1-\beta)}\right| + Y_s^{\beta/(1-\beta)}\left|f(Y_t) - f(Y_s)\right| \\ &\leqslant \sup_{0\leqslant t\leqslant T}\left|f(Y_t)\right| \frac{\beta}{1-\beta} \sup_{0\leqslant t\leqslant T} Y_t^{(2\beta-1)/(1-\beta)} |Y_t - Y_s| \\ &\quad + \sup_{0\leqslant t\leqslant T} Y_t^{\beta/(1-\beta)} \sup_{0\leqslant t\leqslant T}\left|f'(Y_t)\right||Y_t - Y_s| = O_\omega\left(n^{-\gamma}\right).\end{aligned}$$

Thus,

$$\int_{t_k^n}^{t_{k+1}^n} \left|g(X_s) - g(X_{t_k^n})\right| \mathrm{d}s = O_\omega\left(n^{-1-\gamma}\right), \qquad \int_{t_{i-1}^n}^{t_i^n} \left|g(X_s) - g(X_{t_k^n})\right| \mathrm{d}s = O_\omega\left(n^{-1-\gamma}\right).$$

Moreover, by Love–Young inequality, (10) and Hölder continuity of B^H we get

$$\left|\int_{t_{i-1}^n}^{t_i^n} \left[X_s^\beta - X_{t_i^n}^\beta\right] B_s^H\right| \leqslant C_{\gamma,\gamma} \frac{\beta}{(1-\beta)} \sup_{0\leqslant t\leqslant T} Y_t^{(2\beta-1)/(1-\beta)} K_{Y,T} G_{\gamma,T} n^{-2\gamma},$$

where $K_{Y,T} = \sup_{s,t\in[0,T],\, s\neq t} |Y_t - Y_s|/|s-t|^\gamma < \infty$ a.s. Thus, we get the statement of the lemma. □

Proof of Theorem 2. Since X is a positive solution, it follows from Lemma 1 and (A.1) that

$$\begin{aligned}\widetilde{V}_{n,T}^{(2)X} &= \sum_{k=1}^{n-1} \left(\sigma \Delta_{n,k}^{(2)} B^H + O_\omega\left(n^{-2\gamma}\right)\right)^2 = \sigma^2 \sum_{k=1}^{n-1} \left(\Delta_{n,k}^{(2)} B^H\right)^2 + O_\omega\left(n^{1-3\gamma}\right) \\ &= \frac{\sigma^2 T^{2H}(4-2^{2H})}{n^{2H-1}} V_{n,T}^{(2)\widehat{B}^H} + O_\omega\left(n^{1-3\gamma}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sigma^2 T^{2H}(4-2^{2H})}{n^{2H-1}}\left[1+O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right)\right] + O_\omega\left(n^{1-3\gamma}\right)\\
&= \frac{\sigma^2 T^{2H}(4-2^{2H})}{n^{2H-1}}\left[1+O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right) + O_\omega\left(n^{-\gamma+2(H-\gamma)}\right)\right]\\
&= \frac{\sigma^2 T^{2H}(4-2^{2H})}{n^{2H-1}}\left[1+O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right)\right] \qquad (11)
\end{aligned}
$$

if γ is slightly different from H. Thus, by Maclaurin's expansion

$$
\begin{aligned}
\widehat{H}_n &= \frac{1}{2} - \frac{1}{2\ln 2}\ln\frac{\frac{T^{2H}(4-2^{2H})}{(2n)^{2H-1}}[1+O_\omega((\frac{\ln n}{n})^{1/2})]}{\frac{T^{2H}(4-2^{2H})}{n^{2H-1}}[1+O_\omega((\frac{\ln n}{n})^{1/2})]}\\
&= \frac{1}{2} - \frac{1}{2\ln 2}\ln\left(\frac{1}{2^{2H-1}}\left(\frac{1+O_\omega((\frac{\ln n}{n})^{1/2})}{1+O_\omega((\frac{\ln n}{n})^{1/2})}\right)\right)\\
&= H + \ln\left(1+O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right)\right) = H + O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right).
\end{aligned}
$$

Repeating the proof of Theorem 3.17 in [5] and using (11), we get

$$
\begin{aligned}
\widehat{H}_n &= \frac{1}{2} - \frac{1}{2\ln 2}\ln\frac{\frac{\sigma^2 T^{2H}(4-2^{2H})}{(2n)^{2H-1}}V_{2n,T}^{(2)\widehat{B}^H} + O_\omega(n^{1-3\gamma})}{\frac{\sigma^2 T^{2H}(4-2^{2H})}{n^{2H-1}}V_{n,T}^{(2)\widehat{B}^H} + O_\omega(n^{1-3\gamma})}\\
&= \frac{1}{2} - \frac{1}{2\ln 2}\ln\left(\frac{V_{2n,T}^{(2)\widehat{B}^H}}{2^{2H-1}V_{n,T}^{(2)\widehat{B}^H}}\left(\frac{1+O_\omega(n^{-\gamma+2(H-\gamma)})}{1+O_\omega(n^{-\gamma+2(H-\gamma)})}\right)\right)\\
&= \widetilde{H}_n - \frac{1}{2\ln 2}\ln\left(1+O_\omega\left(n^{-\gamma+2(H-\gamma)}\right)\right) = \widetilde{H}_n + O_\omega\left(n^{-\gamma+2(H-\gamma)}\right),
\end{aligned}
$$

where

$$
\widetilde{H}_n = \frac{1}{2} - \frac{1}{2\ln 2}\ln\frac{V_{2n,T}^{(2)\widehat{B}^H}}{2^{2H-1}V_{n,T}^{(2)\widehat{B}^H}} = H - \frac{1}{2\ln 2}\ln\frac{V_{2n,T}^{(2)\widehat{B}^H}}{V_{n,T}^{(2)\widehat{B}^H}}.
$$

Applying the limit results of Section A.2, we obtain $\widetilde{H}_n \to H$ a.s. and

$$
2\ln 2\sqrt{n}(\widetilde{H}_n - H) \xrightarrow{\mathrm{d}} \mathcal{N}\left(0,\sigma_H^2\right).
$$

Now, to finish the proof, it is enough to apply the Slutsky's theorem and the results obtained above. Note that the limit variance σ_H^2 of $\widehat{H}_n$ equals that to $\widetilde{H}_n$. □

Proof of Theorem 3. First, we prove (4). By Remark 1 the values of the approximation $(\widehat{Y}_k)$ are strictly positive for $y_0 > 0$ and $0 < h < h_0$.

By definition of $\widehat{Y}^n$, for any $t \in (t_k^n, t_{k+1}^n]$,

$$\begin{aligned}
Y_t - \widehat{Y}_t^n &= Y_t - \frac{t - t_k^n}{h}\,\widehat{Y}_{k+1} - \frac{t_{k+1}^n - t}{h}\,\widehat{Y}_k \\
&= \frac{t_k^n - t}{h}\,(1-\beta)\Biggl[\int_t^{t_{k+1}^n} f(Y_s)\,\mathrm{d}s + \sigma\bigl(B^H_{t_{k+1}^n} - B^H_t\bigr)\Biggr] \\
&\quad + \frac{t_{k+1}^n - t}{h}\,(1-\beta)\Biggl[\int_{t_k^n}^{t} f(Y_s)\,\mathrm{d}s + \sigma\bigl(B^H_t - B^H_{t_k^n}\bigr)\Biggr] \\
&\quad + \frac{t - t_k^n}{h}(Y_{t_{k+1}^n} - \widehat{Y}_{k+1}) + \frac{t_{k+1}^n - t}{h}(Y_{t_k^n} - \widehat{Y}_k).
\end{aligned}$$

The asymptotic behavior of the first two terms is $O_\omega(n^{-\gamma})$. Thus, it remains for us to obtain the asymptotic of the last two terms.

Note that

$$\begin{aligned}
Y_{t_{k+1}^n} - \widehat{Y}_{n,k+1} &= Y_{t_k^n} - \widehat{Y}_{n,k} + (1-\beta)\int_{t_k^n}^{t_{k+1}^n}\bigl[f(Y_s) - f(\widehat{Y}_{n,k+1})\bigr]\,\mathrm{d}s \\
&= Y_{t_k^n} - \widehat{Y}_{n,k} + (1-\beta)\int_{t_k^n}^{t_{k+1}^n}\bigl[f(Y_s) - f(Y_{t_{k+1}^n})\bigr]\,\mathrm{d}s \\
&\quad + (1-\beta)\zeta_{k+1}(Y_{t_{k+1}^n} - \widehat{Y}_{n,k+1})h,
\end{aligned}$$

where $\zeta_{k+1} = f'(Y_{t_{k+1}^n} + \theta(\widehat{Y}_{n,k+1} - Y_{t_{k+1}^n}))$, $\theta \in (0,1)$. Then

$$\begin{aligned}
Y_{t_{k+1}^n} - \widehat{Y}_{n,k+1} &= \frac{1}{1-(1-\beta)\zeta_{k+1}h}\Biggl[Y_{t_k^n} - \widehat{Y}_{n,k} + (1-\beta)\int_{t_k^n}^{t_{k+1}^n}\bigl[f(Y_s) - f(Y_{t_{k+1}^n})\bigr]\,\mathrm{d}s\Biggr] \\
&= \sum_{i=1}^{k+1} I_i \prod_{j=i}^{k+1}\bigl(1 - \zeta_j(1-\beta)h\bigr)^{-1}, \qquad (12)
\end{aligned}$$

where $I_i = (1-\beta)\int_{t_{i-1}^n}^{t_i^n}[f(Y_s) - f(Y_{t_i^n})]\,\mathrm{d}s$.

Note that $1 - \zeta_i(1-\beta)h \geqslant 1 - K^+h > 0$ for small h since $f'(x) \leqslant K$, where $K^+ = \max\{0, K\}$. Applying inequality $\ln(1/(1-x)) \leqslant x/(1-x)$, $x < 1$, we get

$$\begin{aligned}
\prod_{j=i}^{k+1}\bigl(1 - \zeta_j(1-\beta)h\bigr)^{-1} &\leqslant \bigl(1 - K^+h\bigr)^{-(k+1-i)} \leqslant \mathrm{e}^{n\ln(1/(1-K^+h)} \\
&\leqslant \mathrm{e}^{n(K^+h)/(1-K^+h)} \leqslant \mathrm{e}^{(K^+T)/(1-K^+h)}.
\end{aligned}$$

Continuity of the function f' and the positivity of the process $Y \in \mathcal{C}^\gamma([0,T])$, $\gamma \in (1/2, H)$, gives an estimate

$$\left|\int_{t_{k-1}^n}^{t_k^n} \big[f(Y_s) - f(Y_{t_k^n})\big]\,\mathrm{d}s\right| \leqslant h \max_{0\leqslant t\leqslant T}\big|f'(Y_t)\big| \max_{t_{k-1}^n\leqslant t\leqslant t_k^n}\big|Y_t - Y_{t_k^n}\big|$$
$$= O_\omega\big(h^{1+\gamma}\big). \tag{13}$$

From (12) it follows that

$$\max_{1\leqslant k\leqslant n} |Y_{t_k^n} - \widehat{Y}_{n,k}| = O_\omega\big(n^{-\gamma}\big). \tag{14}$$

This finishes the proof of (4).

It remains to prove (5). By applying inequality

$$x^p - y^p < px^{p-1}(x-y), \quad 0<y<x,\ p>1,$$

it follows that

$$\big|x^p - y^p\big| < p|x-y|\big(\max\big\{|x|,|y|\big\}\big)^{p-1}.$$

Since $X_t = Y_t^{1/(1-\beta)}$, then

$$\sup_{0\leqslant t\leqslant T}\big|X_t - \big(\widehat{Y}_t^n\big)^{1/(1-\beta)}\big|$$
$$= \sup_{0\leqslant t\leqslant T}\big|Y_t^{1/(1-\beta)} - \big(\widehat{Y}_t^n\big)^{1/(1-\beta)}\big|$$
$$\leqslant \frac{1}{1-\beta}\sup_{0\leqslant t\leqslant T}\big|Y_t - \widehat{Y}_t^n\big|\Big(\max\Big\{\sup_{0\leqslant t\leqslant T}|Y_t|, \sup_{0\leqslant t\leqslant T}\big|\widehat{Y}_t^n\big|\Big\}\Big)^{\beta/(1-\beta)}.$$

From (4) and finiteness of $\sup_{0\leqslant t\leqslant T}|Y_t|$ we have

$$\sup_{0\leqslant t\leqslant T}\big|\widehat{Y}_t^n\big| \leqslant \sup_{0\leqslant t\leqslant T}\big|\widehat{Y}_t^n - Y_t\big| + \sup_{0\leqslant t\leqslant T}|Y_t| \leqslant O_\omega\big(n^{-\gamma}\big) + \sup_{0\leqslant t\leqslant T}|Y_t|$$
$$= O_\omega(1).$$

Thus,

$$\sup_{0\leqslant t\leqslant T}\big|X_t - \big(\widehat{Y}_t^n\big)^{1/(1-\beta)}\big| = O_\omega\big(n^{-\gamma}\big).$$

This finishes the proof of (5). □

To prove Theorem 4, we need the following statement.

Lemma 2. *Let* $1/2 \leqslant \beta < 1$ *and* $\beta = (m-1)/m$, $m \in \mathbb{N}$. *If conditions of Theorem* 3 *are satisfied, then*

$$\Delta_{n,i}^{(2)}\widehat{Y}^m = (1-\beta)\sigma\widehat{Y}_{n,i}^{m-1}\Delta_{n,i}^{(2)}B^H + O_\omega\big(n^{-2\gamma}\big).$$

Proof. We now recall a well-known equality of algebra

$$a^m - b^m = (a-b)\sum_{k=0}^{m-1} a^{m-1-k}b^k.$$

From this equality we get

$$\begin{aligned}
a^m - b^m &= (a-b)\left[a^{m-2}(a-b) + 2a^{m-2}b + \sum_{k=2}^{m-1} a^{m-1-k}b^k\right] \\
&= (a-b)\left[a^{m-2}(a-b) + 2a^{m-2}b + a^{m-3}b^2 + \sum_{k=3}^{m-1} a^{m-1-k}b^k\right] \\
&= (a-b)\left[a^{m-2}(a-b) + 2a^{m-3}b(a-b) + 3a^{m-3}b^2 + \sum_{k=3}^{m-1} a^{m-1-k}b^k\right] \\
&= (a-b)\left[(a-b)\sum_{k=1}^{m-1} ka^{m-1-k}b^{k-1} + mb^{m-1}\right], \\
a^m - b^m &= (a-b)\left[b^{m-2}(b-a) + 2ab^{m-2} + \sum_{k=2}^{m-1} a^k b^{m-1-k}\right] \\
&= (a-b)\left[(b-a)\sum_{k=1}^{m-1} ka^{k-1}b^{m-1-k} + ma^{m-1}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta_{n,i}^{(2)}\widehat{Y}^m &= (\widehat{Y}_{n,i+1} - \widehat{Y}_{n,i})^2 \sum_{k=1}^{m-1} k\widehat{Y}_{n,i+1}^{m-1-k}\widehat{Y}_{n,i}^{k-1} + m\widehat{Y}_{n,i}^{m-1}(\widehat{Y}_{n,i+1} - \widehat{Y}_{n,i}) \\
&\quad + (\widehat{Y}_{n,i-1} - \widehat{Y}_{n,i})^2 \sum_{k=1}^{m-1} k\widehat{Y}_{n,i}^{k-1}\widehat{Y}_{n,i-1}^{m-1-k} - m\widehat{Y}_{n,i}^{m-1}(\widehat{Y}_{n,i} - \widehat{Y}_{n,i-1}) \\
&= m\widehat{Y}_{n,i}^{m-1}\Delta_{n,i}^{(2)}\widehat{Y} + (\widehat{Y}_{n,i+1} - \widehat{Y}_{n,i})^2 \sum_{k=1}^{m-1} k\widehat{Y}_{n,i}^{m-1-k}\widehat{Y}_{n,i-1}^{k-1} \\
&\quad + (\widehat{Y}_{n,i-1} - \widehat{Y}_{n,i})^2 \sum_{k=1}^{m-1} k\widehat{Y}_{n,i}^{k-1}\widehat{Y}_{n,i-1}^{m-1-k}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{m-1} k\left|\widehat{Y}_{n,i}^{m-1-k}\widehat{Y}_{n,i-1}^{k-1}\right| \leqslant \max_{1\leqslant i\leqslant n}|\widehat{Y}_{n,i}|^{m-2}\sum_{k=1}^{m-1} k \leqslant \frac{m^2}{2}\max_{1\leqslant i\leqslant n}|\widehat{Y}_{n,i}|^{m-2}, \\
&\sum_{k=1}^{m-1} k\left|\widehat{Y}_{n,i}^{k-1}\widehat{Y}_{n,i-1}^{m-1-k}\right| \leqslant \frac{m^2}{2}\max_{1\leqslant i\leqslant n}|\widehat{Y}_{n,i}|^{m-2}.
\end{aligned}$$

Since from Theorem 3 we have

$$|\widehat{Y}_{n,i+1} - \widehat{Y}_{n,i}| = O_\omega\big(n^{-\gamma}\big)$$

and

$$|\widehat{Y}_{n,i}| \leqslant |\widehat{Y}_{n,i} - Y_{t_i^n}| + |Y_{t_i^n}| \leqslant O_\omega\big(n^{-\gamma}\big) + \sup_{0\leqslant t\leqslant T}|Y_t| = O_\omega(1),$$

then

$$\Delta_{n,i}^{(2)}\widehat{Y}^m = m\widehat{Y}_{n,i}^{m-1}\Delta_{n,i}^{(2)}\widehat{Y} + O_\omega\big(n^{-2\gamma}\big).$$

To prove the assertion of the lemma, it remains to prove that

$$\Delta_{n,i}^{(2)}\widehat{Y} = \Delta_{n,i}^{(2)}Y + O_\omega\big(n^{-2\gamma}\big) \tag{15}$$

since

$$\Delta_{n,i}^{(2)}Y = (1-\beta)\sigma\Delta_{n,i}^{(2)}B^H + O_\omega\big(n^{-1-\gamma}\big).$$

The last equality follows from (13). Indeed, applying formula (12), we get

$$\begin{aligned}
&\Delta_{n,i+1}Y - \Delta_{n,i+1}\widehat{Y} \\
&\quad= \sum_{k=1}^{i+1} I_k \prod_{j=k}^{i+1}\big(1-\zeta_j(1-\beta)h\big)^{-1} - \sum_{k=1}^{i} I_k \prod_{j=k}^{i}\big(1-\zeta_j(1-\beta)h\big)^{-1} \\
&\quad= I_{i+1}\big(1-\zeta_{i+1}(1-\beta)h\big)^{-1} \\
&\qquad+ \sum_{k=1}^{i} I_k\left(\prod_{j=k}^{i+1}\big(1-\zeta_j(1-\beta)h\big)^{-1} - \prod_{j=k}^{i}\big(1-\zeta_j(1-\beta)h\big)^{-1}\right) \\
&\quad= I_{i+1}\big(1-\zeta_{i+1}(1-\beta)h\big)^{-1} \\
&\qquad+ \big(\big(1-\zeta_{i+1}(1-\beta)h\big)^{-1} - 1\big)\sum_{k=1}^{i} I_k \prod_{j=k}^{i}\big(1-\zeta_j(1-\beta)h\big)^{-1} \\
&\quad= I_{i+1}\big(1-\zeta_{i+1}(1-\beta)h\big)^{-1} + \big(\big(1-\zeta_{i+1}(1-\beta)h\big)^{-1} - 1\big)(Y_{t_i^n} - \widehat{Y}_{n,i}) \\
&\quad= \big(1-\zeta_{i+1}(1-\beta)h\big)^{-1}\big[I_{i+1} + \zeta_{i+1}(1-\beta)h(Y_{t_i^n} - \widehat{Y}_{n,i})\big],
\end{aligned}$$

where $\Delta_{n,i+1}\widehat{Y} = \widehat{Y}_{n,i+1} - \widehat{Y}_{n,i}$. By Remark 1 the values of the approximation $(\widehat{Y}_k)$ are strictly positive for small h. Since Y is continuous and $Y_t > 0$ for all $t \in [0,T]$, then from (14) it follows that

$$\begin{aligned}
&Y_{t_{k+1}^n} + \theta(\widehat{Y}_{n,k+1} - Y_{t_{k+1}^n}) \\
&\qquad\geqslant (1-\theta)\inf_{0\leqslant t\leqslant T} Y_t + \theta\widehat{Y}_{n,k+1} \geqslant (1-\theta)\inf_{0\leqslant t\leqslant T} Y_t > 0, \\
&Y_{t_{k+1}^n} + \theta(\widehat{Y}_{n,k+1} - Y_{t_{k+1}^n}) \\
&\qquad\leqslant \sup_{0\leqslant t\leqslant T} Y_t + O_\omega\big(n^{-\gamma}\big) \leqslant \sup_{0\leqslant t\leqslant T} Y_t + O_\omega(1).
\end{aligned}$$

Thus,

$$0 < \inf_{n\geqslant 2} \min_{1\leqslant k\leqslant n-1} f'\bigl(Y_{t^n_{k+1}} + \theta(\widehat{Y}_{n,k+1} - Y_{t^n_{k+1}})\bigr) \leqslant \sup_{n\geqslant 2} \max_{1\leqslant k\leqslant n-1} f'\bigl(Y_{t^n_{k+1}} + \theta(\widehat{Y}_{n,k+1} - Y_{t^n_{k+1}})\bigr) < \infty \quad \text{a.s.}$$

and therefore

$$\zeta_{i+1}(1-\beta)h(Y_{t^n_i} - \widehat{Y}_{n,i}) = O_\omega\bigl(h^{1+\gamma}\bigr). \tag{16}$$

Since for small h, it follows that $\zeta_i(1-\beta)h \leqslant Kh < 1$, then

$$0 < \bigl(1 - \zeta_{i+1}(1-\beta)h\bigr)^{-1} \leqslant \bigl(1 - K^+h\bigr)^{-1}. \tag{17}$$

From (13), (16) and (17) it follows that

$$\Delta_{n,i+1}Y - \Delta_{n,i+1}\widehat{Y} = O_\omega\bigl(h^{1+\gamma}\bigr).$$

So, we obtain (15). □

Proof of Theorem 4. The proof is a similar to that of Theorem 2. □

5 Examples

Example 1. The fractional CKLS model has a unique positive solution, and Theorem 3 holds.

Proof. The SDE, which we call a fractional CKLS model, has the form

$$X_t = x_0 + \int_0^t (a_1 - a_2X_s)\,\mathrm{d}s + \sigma \int_0^t X_s^\beta\,\mathrm{d}B_s^H, \quad \frac{1}{2} \leqslant \beta < 1,$$

with the initial value $x_0 > 0$, where $H \in (1/2, 1)$, deterministic constants $a_1 > 0$, $a_2 \in \mathbb{R}$ and $\sigma > 0$.

Set $f(x) = a_1/x^{\beta/(1-\beta)} - a_2x$. Note that

$$\lim_{x\to 0+}\biggl(f(x) - \frac{a_1}{2x^{\beta/(1-\beta)}}\biggr) = +\infty$$

with $a = a_1/2$, $\alpha = (2\beta-1)/(1-\beta) \geqslant 0$. Thus, conditions (C1) and (C2) are satisfied. Moreover, since $g(x) = x^\beta f(x^{1-\beta}) = a_1 - a_2x$, then from Theorem 1 we get that the CKLS model has a unique positive solution.

We verify condition (C4). The function

$$F(x) = x - (1-\beta)f(x)h = -\frac{a_1(1-\beta)}{x^{\beta/(1-\beta)}}\,h + \bigl(1 + a_2(1-\beta)h\bigr)x$$

is continuous on $(0,\infty)$. It is clear that $1+a_2(1-\beta)h>0$ for any $h>0$ if $a_2\geqslant 0$. If $a_2<0$ and the sequence of uniform partitions π of the interval $[0,T]$ is such that $h<1/(-a_2(1-\beta))$, then $1+a_2(1-\beta)h>0$. Set $h_0=1/(-a_2(1-\beta))$. Thus, $\lim_{x\to 0^+}F(x)=-\infty$ and $\lim_{x\to\infty}F(x)=+\infty$ for $h\in(0,h_0)$. Moreover, since

$$F'(x)=\frac{a_1\beta}{x^{1/(1-\beta)}}\,h+\bigl(1+a_2(1-\beta)h\bigr),$$

then the function F is strictly increasing for $h\in(0,h_0)$. Thus, condition (C4) is satisfied.

Since the function $f'(x)$ is continuous on $(0,\infty)$ and

$$f'(x)=-\frac{a_1\beta}{1-\beta}\,x^{-1/(1-\beta)}-a_2\leqslant|a_2|,$$

then condition (C3) is fulfilled and Theorem 3 holds. □

Example 2. The Ait-Sahalia model has a unique positive solution, and Theorem 3 holds.

Proof. The SDE, which we call a fractional Ait-Sahalia type model, has the form

$$X_t=x_0+\int_0^t\bigl(a_1X_s^{-1}-a_2+a_3X_s-a_4X_s^r\bigr)\,\mathrm{d}s+\sigma\int_0^t X_s^\beta\,\mathrm{d}B_s^H$$

with the initial value $x_0>0$, $r\in(-1,1)$, where $H\in(1/2,1)$, $1/2\leqslant\beta<1$, deterministic constants $a_1,a_2,a_3,a_4\geqslant 0$ and $\sigma>0$.

Set

$$f(x)=\frac{a_1}{x^{(1+\beta)/(1-\beta)}}-\frac{a_2}{x^{\beta/(1-\beta)}}+a_3x-\frac{a_4}{x^{(\beta-r)/(1-\beta)}}.$$

Note that

$$\begin{aligned}f(x)&-\frac{a_1}{2x^{(1+\beta)/(1-\beta)}}\\&=x^{-(1+\beta)/(1-\beta)}\biggl(\frac{a_1}{2}-a_2x^{1/(1-\beta)}-a_4x^{(r+1)/(1-\beta)}\biggr)+a_3x\end{aligned}$$

and the term in the brackets is positive for small x. Thus,

$$\lim_{x\to 0^+}\biggl(f(x)-\frac{a_1}{2x^{(1+\beta)/(1-\beta)}}\biggr)=+\infty$$

with $a=a_1/2$, $\alpha=(2\beta)/(1-\beta)\geqslant 2$. Therefore, conditions (C1) and (C2) are satisfied. Moreover, since $g(x)=x^\beta f(x^{1-\beta})=a_1x^{-1}-a_2+a_3x-a_4x^r$, then from Theorem 1 we get that the Ait-Sahalia model for $1/2\leqslant\beta<1$, $r\in(-1,1)$, has a unique positive solution.

Now we verify condition (C4). Clearly, the function

$$F(x) = x - (1-\beta)f(x)h \\ = x\bigl(1-a_3(1-\beta)h\bigr) - (1-\beta)\left(\frac{a_1}{x^{(1+\beta)/(1-\beta)}} - \frac{a_2}{x^{\beta/(1-\beta)}} - \frac{a_4}{x^{(\beta-r)/(1-\beta)}}\right)h$$

is continuous on $(0,\infty)$. Assume, that $h_0 = 1/(a_3(1-\beta))$. Then for $h \in (0, h_0)$, we have $1 - a_3(1-\beta)h > 0$ and $\lim_{x\to 0+} F(x) = -\infty$, $\lim_{x\to\infty} F(x) = +\infty$.

Note that

$$f'(x) = -a_1\frac{1+\beta}{1-\beta}x^{-2/(1-\beta)} + a_2\frac{\beta}{1-\beta}x^{-1/(1-\beta)} + a_3 + a_4\frac{\beta-r}{1-\beta}x^{-(1-r)/(1-\beta)}.$$

Since the derivative $f'(x)$ is continuous on $(0,\infty)$ and $\lim_{x\to 0+} f'(x) = -\infty$, $\lim_{x\to\infty} f'(x) = a_3$, then there is a constant K such that $f'(x) \leqslant K$. Therefore,

$$(x-y)\bigl(F(x)-F(y)\bigr) = (x-y)^2 - (1-\beta)(x-y)\bigl(f(x)-f(y)\bigr)h \\ \geqslant \bigl(1-K^+h\bigr)(x-y)^2 > 0$$

with $K^+ = \max\{0, K\}$ and the strict monotonicity is obtained. Therefore, the conditions of Theorem 3 are satisfied. □

6 Simulation results

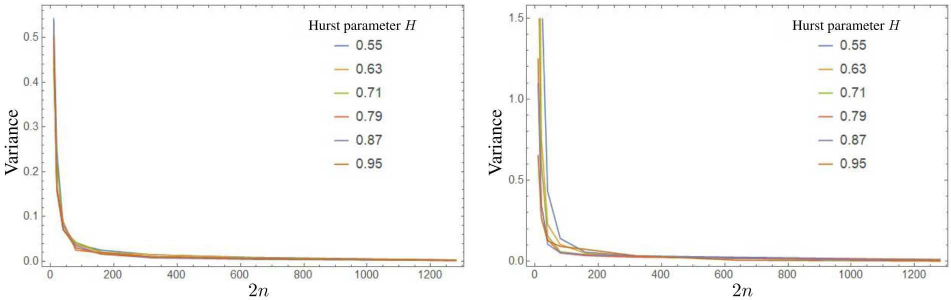
The purpose of this section is to provide some simulations in order to illustrate various aspects of the suggested estimator. We consider CKLS model. The simulation of the obtained estimate presented below was performed using *Wolfram Mathematica*. The values of the constants involved in these simulations were $x_0 = 4$, $a = 1$, $b = 2$, $\sigma = 1$. We considered these sample paths on the unit interval, hence, $T = 1$. The number of batches were 200 in all of the considered cases.

The CKLS model after Lamperti transform has the form

$$Y_t = x_0 + (1-\beta)\int_0^t \left(\frac{a_1}{Y_s^{\beta/(1-\beta)}} - a_2 Y_s\right) \mathrm{d}s + \sigma(1-\beta)\int_0^t Y_s^\beta \,\mathrm{d}B_s^H, \quad \frac{1}{2} \leqslant \beta < 1.$$

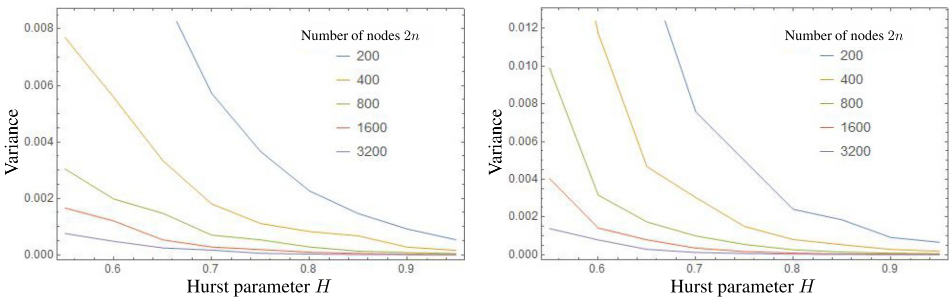
Applying Theorem 4, we calculate the estimator $\widehat{H}_n^E$ with $\beta = (m-1)/m$, $m \in \mathbb{N}$. The asymptotic behavior of the variance of the difference $H - \widehat{H}_n^E$ for different m, n, and H is shown in Figs. 1, 2.

Figure 1 shows that the variance of the difference $H - \widehat{H}_n^E$ decreases as the sample size increases. Figure 2 shows how the variance of the difference $H - \widehat{H}_n^E$ for different H depends on the sample size. We see that with increasing sample size, the variance decreases for all H values.



(a) $m = 2$ (b) $m = 8$

Figure 1. Dependence of variance on n.



(a) $m = 2$ (b) $m = 8$

Figure 2. Dependence of variance on H.

Appendix

The following facts are taken from the book [5].

A.1 Love–Young inequality

For any $a < b$, $C^\gamma([a,b])$ denotes the space of Hölder continuous functions of order $\gamma > 0$ on $[a,b]$. Let $f \in C^\lambda([a,b])$ and $g \in C^\mu([a,b])$ with $\lambda + \mu > 1$ and

$$K_f = \sup_{\substack{s,t\in[a,b]\\ s\neq t}} \frac{|f(t) - f(s)|}{|s-t|^\lambda}, \qquad K_g = \sup_{\substack{s,t\in[a,b]\\ s\neq t}} \frac{|g(t) - g(s)|}{|s-t|^\mu}.$$

Love–Young inequality has the form: for any $y \in [a,b]$,

$$\left| \int_a^b f\,\mathrm{d}g - f(y)\big[g(b) - g(a)\big] \right| \leqslant C_{\mu,\lambda} K_f K_g (b-a)^{\lambda+\mu}, \quad C_{\mu,\lambda} = \zeta(\mu+\lambda),$$

where $\zeta(s)$ denotes the Riemann zeta function, i.e., $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$ [5, p. 10].

A.2 Several results on fBm

Recall that fBm $B^H = \{B_t^H, t \geqslant 0\}$ with the Hurst index $H \in (0,1)$ is a real-valued continuous centered Gaussian process with covariance given by

$$\mathbf{E}\left(B_t^H B_s^H\right) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

For consideration of strong consistency and asymptotic normality of the given estimators, we need several facts regarding B^H.

Limit results. Let

$$V_{n,T}^{(2)\widehat{B}^H} = \frac{n^{2H-1}}{T^{2H}(4-2^{2H})} \sum_{k=1}^{n-1} \left(\Delta_{n,k}^{(2)} B^H\right)^2, \quad H \neq \frac{1}{2}.$$

Then (see [4], [5, pp. 46, 52, 58, 66]) $V_{n,T}^{(2)\widehat{B}^H} \to 1$ a.s. as $n \to \infty$ and

$$\sqrt{n}\begin{pmatrix} V_{n,T}^{(2)\widehat{B}^H} - 1 \\ V_{2n,T}^{(2)\widehat{B}^H} - 1 \end{pmatrix} \xrightarrow{\mathrm{d}} \mathcal{N}(0;\Sigma_H), \quad \Sigma_H = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix},$$

where $\mathcal{N}(0;\Sigma_H)$ is a Gaussian vector with

$$\Sigma_{11} = 2\left(1 + \frac{2}{(4-2^{2H})^2}\sum_{j=1}^{\infty} \widehat{\rho}_H^2(j)\right), \qquad \Sigma_{22} = \frac{1}{2}\Sigma_{11},$$

$$\Sigma_{12} = \Sigma_{21} = \frac{1}{2^{2H}(4-2^{2H})^2}\sum_{j\in\mathbb{Z}} \widetilde{\rho}_H^2(j),$$

$$\widehat{\rho}_H(j) = \frac{1}{2}\left[-6|j|^{2H} - |j-2|^{2H} - |j+2|^{2H} + 4|j-1|^{2H} + 4|j+1|^{2H}\right],$$

$$\widetilde{\rho}_H(j) = \frac{1}{2}\left[|j+1|^{2H} + 2|j+2|^{2H} - |j+3|^{2H} + |j-1|^{2H} - 4|j|^{2H} - |j-3|^{2H} + 2|j-2|^{2H}\right].$$

Moreover,

$$V_{n,T}^{(2)\widehat{B}^H} = 1 + O_\omega\left(n^{-1/2}\ln^{1/2} n\right) \tag{A.1}$$

and

$$\sqrt{n}\ln\frac{V_{2n,T}^{\widehat{B}^H}}{V_{n,T}^{\widehat{B}^H}} \xrightarrow{\mathrm{d}} \mathcal{N}\left(0,\sigma_H^2\right)$$

with $\sigma_H^2 = (3/2)\Sigma_{11} - 2\Sigma_{12}$.

Hölder-continuity of B^H. It is known that almost all sample paths of an fBm B^H are locally Hölder of order strictly less than H, $H \in (0,1)$. To be more precise, for all

$T > 0$, there exists a nonnegative random variable $G_{\gamma,T}$ such that $\mathbf{E}(|G_{\gamma,T}|^p) < \infty$ for all $p \geqslant 1$, and

$$\left|B_t^H - B_s^H\right| \leqslant G_{\gamma,T}|t-s|^\gamma \quad \text{a.s.}$$

for all $s, t \in [0, T]$, where $\gamma \in (0, H)$ (see [5, p. 4]).

References

1. A. Alfonsi, Strong order one convergence of a drift implicit Euler scheme: Application to the CIR process, *Stat. Probab. Lett.*, **83**(2):602–607, 2013.
2. N.T. Dung, Tail probabilities of solutions to a generalized Ait-Sahalia interest rate model, *Stat. Probab. Lett.*, **112**:98–104, 2016.
3. Y. Hu, D. Nualart, X. Song, A singular stochastic differential equation driven by fractional Brownian motion, *Stat. Probab. Lett.*, **78**(14):2075–2085, 2008.
4. K. Kubilius, CLT for quadratic variation of Gaussian processes and its application to the estimation of the Orey index, *Stat. Probab. Lett.*, **165**:108845, 2020.
5. K. Kubilius, Yu. Mishura, K. Ralchenko, *Parameter Estimation in Fractional Diffusion Models*, Bocconi Springer Ser., Vol. 8, Springer, 2017.
6. K. Kubilius, V. Skorniakov, On some estimators of the hurst index of the solution of SDE driven by a fractional Brownian motion, *Stat. Probab. Lett.*, **109**:159–167, 2016.
7. K. Kubilius, V. Skorniakov, D. Melichov, Estimation of parameters of SDE driven by fractional Brownian motion with polynomial drift, *J. Stat. Comput. Simulation*, **86**(10):1954–1969, 2016.
8. E. Lépinette, F. Mehrdoust, A fractional version of the Heston model with Hurst parameter $H \in (1/2, 1)$, *Dyn. Syst. Appl.*, **26**(3–4):535–548, 2017.
9. N. Marie, A generalized mean-reverting equation and applications, *ESAIM, Probab. Stat.*, **18**:799–828, 2014.
10. A. Melnikov, Y. Mishura, G. Shevchenko, Stochastic viability and comparison theorems for mixed stochastic differential equations, *Methodol. Comput. Appl. Probab.*, **17**(1):169–188, 2015.
11. Y. Mishura, V. Piterbarg, K. Ralchenko, A. Yurchenko-Tytarenko, Stochastic representation and pathwise properties of fractional Cox–Ingersoll–Ross process, *Theory Probab. Math. Stat.*, **97**:167–182, 2018.
12. Y. Mishura, A. Yurchenko-Tytarenko, Fractional Cox–Ingersoll–Ross process with non-zero "mean", *Mod. Stoch., Theory Appl.*, **5**(1):99–111, 2018.
13. A. Neuenkirch, L. Szpruch, First order strong approximations of scalar SDEs defined in a domain, *Numer. Math.*, **128**(1):103–136, 2014.
14. L. Szpruch, X. Mao, D.J. Higham, J. Pan, Numerical simulation of a strongly nonlinear Ait-Sahalia-type interest rate model, *BIT*, **51**(2):405–425, 2011.
15. F. Wu, X. Mao, K. Chen, A highly sensitive mean-reverting process in finance and the Euler–Maruyama approximations, *J. Math. Anal. Appl.*, **348**(1):540–554, 2008.
16. S.-Q. Zhang, C. Yuan, Stochastic differential equations driven by fractional Brownian motion with locally Lipschitiz drift and their Euler approximation, arXiv:1812.11382.