

DISCRETE UNIVERSALITY OF THE RIEMANN ZETA-FUNCTION IN SHORT INTERVALS

Antanas Laurinčikas

We consider the approximation of analytic functions by shifts of the Riemann zeta-function $\zeta(s + ikh)$ with fixed $h > 0$ when positive integers k run over the interval $[N, N + M]$, where $N^{1/3}(\log N)^{26/15} \leq M \leq N$, and prove that those k have a positive lower density as $N \rightarrow \infty$. The same is true for some compositions. Two types of $h > 0$ are discussed separately.

1. INTRODUCTION

The Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, is given, for $\sigma > 1$, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the infinite product is taken over all prime numbers. Moreover, the function $\zeta(s)$ has the analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. The function $\zeta(s)$ is one of the most interesting and mysterious analytic objects, therefore, much attention is devoted to investigations of its value-distribution. One of the most important properties of $\zeta(s)$ is its universality discovered by Voronin in [20]. Roughly speaking, the Voronin theorem says that all analytic non-vanishing functions defined on the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ can be approximated with a given accuracy by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. Moreover, the set of these shifts approximating a given analytic function has a positive lower density. On the other hand, the Voronin

2010 Mathematics Subject Classification. 11M06, 30E10.

Keywords and Phrases. Approximation of Analytic Functions, Discrete Universality, Riemann Zeta-Function, Voronin Theorem.

theorem is noneffective because any τ with approximation property is not known. Voronin understood the effectivization of his theorem as the indication of an interval containing τ with approximation property. The first step in this direction was made by Good in [6]. His general and complicated results in the special case were presented explicitly by Garunkštis in [4]. Finally, in [5], the interval $[T, 2T]$ containing for $T \geq T_0$ a number τ with approximation property was found. Here T_0 is explicitly given, and depends on the approximated function, approximation accuracy as well as on the approximation disc.

Obviously, the interval containing a “good” τ must be short as possible. This raises a problem of universality in short intervals. The first attempt in this direction is given in [10]. Denote by \mathcal{K} the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K . Then, in [10], it was proved that if $T^{1/3}(\log T)^{26/15} \leq H \leq T$, $K \in \mathcal{K}$ and $f(s) \in H_0(K)$, then, for every $\varepsilon > 0$,

$$\liminf_{H \rightarrow \infty} \frac{1}{H} \text{meas} \left\{ \tau \in [T, T+H] : \sup_{s \in K} |\zeta(s+i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{H \rightarrow \infty} \frac{1}{H} \text{meas} \left\{ \tau \in [T, T+H] : \sup_{s \in K} |\zeta(s+i\tau) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The stated above result is of continuous type because τ in shifts $\zeta(s+i\tau)$ can take arbitrary real values. Also, discrete universality theorems are known when τ in approximating shifts takes values from certain discrete sets. Discrete universality theorems for zeta-functions were introduced by Reich in [17]. He proved a discrete universality theorem for Dedekind zeta-functions $\zeta_{\mathbb{K}}(s)$ of number fields \mathbb{K} on the approximation of functions $f(s) \in H_0(K)$ by shifts $\zeta_{\mathbb{K}}(s+ikh)$, $k = 0, 1, \dots$, with every fixed $h > 0$. When $\mathbb{K} = \mathbb{Q}$, we have a discrete universality theorem for the Riemann zeta-function. By a different method, the Reich theorem was obtained by Bagchi in his thesis [1]. More general discrete sets than the arithmetic progression $\{kh\}$ were used in [3], [6], [12] and [9].

The aim of this paper is discrete universality theorems for the Riemann zeta-function in short intervals. Denote by $\#A$ the cardinality of the set A . In what follows, N and M run over positive integers.

Theorem 1. *Suppose that $N^{1/3}(\log N)^{26/15} \leq M \leq N$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \rightarrow \infty} \frac{1}{M+1} \# \left\{ N \leq k \leq N+M : \sup_{s \in K} |\zeta(s+ikh) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{N \rightarrow \infty} \frac{1}{M+1} \# \left\{ N \leq k \leq N+M : \sup_{s \in K} |\zeta(s+ikh) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

We recall that $h > 0$ is an arbitrary fixed number. Of course, the above limits depend on h . Moreover, a certain dependence property of h plays an important role in the proof, and we have to consider two cases separately.

Denote by $H(D)$ the space of analytic functions on the strip D equipped with the topology of uniform convergence on compacta. Theorem 1 can be generalized for compositions of operators in the space $H(D)$ with the function $\zeta(s)$. We will present only one example on the discrete universality of such compositions.

Let a_1, \dots, a_r be distinct complex numbers, and $F : H(D) \rightarrow H(D)$ be an operator. Define the set

$$H_{a_1, \dots, a_r; F}(D) = \{g \in H(D) : g(s) \neq a_j, j = 1, \dots, r\} \cup \{F(0)\}.$$

Moreover, let

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Theorem 2. *Suppose that $N^{1/3}(\log N)^{26/15} \leq M \leq N$, and that $F : H(D) \rightarrow H(D)$ is a continuous operator such that $H_{a_1, \dots, a_r; F}(D) \subset F(S)$. For $r = 1$, let $K \in \mathcal{K}$, and let $f(s)$ be a continuous and $\neq a_1$ function on K which is analytic in the interior of K . For $r \geq 2$, let $K \subset D$ be an arbitrary compact set, and $f(s) \in H_{a_1, \dots, a_r; F}(D)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \rightarrow \infty} \frac{1}{M+1} \# \left\{ N \leq k \leq N+M : \sup_{s \in K} |F(\zeta(s+ikh)) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{N \rightarrow \infty} \frac{1}{M+1} \# \left\{ N \leq k \leq N+M : \sup_{s \in K} |F(\zeta(s+ikh)) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For example, if $r = 1$ and $a_1 = 0$, then Theorem 2 gives the discrete universality for the function $\zeta^n(s)$, $n \in \mathbb{N}$. If $r = 2$ and $a_1 = -1$, $a_2 = 1$, then we obtain the discrete universality for the functions $\sin \zeta(s)$, $\cos \zeta(s)$, $\sinh \zeta(s)$ and $\cosh \zeta(s)$. Actually, suppose, for example, that $F(g) = \sin g$, $g \in H(D)$. We have to show that $F(S) \supset H_{-1,1;F}(D)$. Obviously, $F(0) = 0$. Let f be an arbitrary element of $H_{-1,1;F}(D)$. Solving the equation

$$\frac{e^{ig} - e^{-ig}}{2i} = f,$$

we find

$$g = \frac{1}{i} \log \left(if \pm \sqrt{1 - f^2} \right).$$

Since $f(s) \neq -1$ and 1 , a suitable choice of the logarithm shows that there exists $g \in S$.

Theorem 2 also contains a certain information on the number of zeros of the composition $F(\zeta(s))$.

Theorem 3. Suppose that $N^{1/3}(\log N)^{26/15} \leq M \leq N$, and that $F : H(D) \rightarrow H(D)$ is a continuous operator such that $H_{a_1, \dots, a_r; F}(D) \subset F(S)$, where $\operatorname{Re} a_j \notin (-1/2, 1/2)$, $j = 1, \dots, r$. Then, for every $1/2 < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, F) > 0$ such that, for sufficiently large N , the function $F(\zeta(s + ikh))$ has a zero in the disc

$$\left| s - \frac{\sigma_1 + \sigma_2}{2} \right| \leq \frac{\sigma_2 - \sigma_1}{2}$$

for more than cM numbers k , $N \leq k \leq N + M$.

For the proof of Theorems 1 and 2, we will apply limit theorems in short intervals for probability measures in the space of analytic functions. For these theorems, we need the discrete mean square estimates for the function $\zeta(s)$ over short intervals.

2. MEAN SQUARE ESTIMATES

In this section, we will obtain the estimate for

$$\frac{1}{M} \sum_{k=N}^{N+M} |\zeta(\sigma + ikh + i\tau)|^2$$

with $1/2 < \sigma < 1$, and $\tau \in \mathbb{R}$. We will derive this estimate from the analogical estimate for the continuous mean square. We will use the following result.

Lemma 4. Suppose that $T^{1/3}(\log T)^{26/15} \leq H \leq T$, and that σ , $1/2 < \sigma < 1$, is fixed. Then, for $\tau \in \mathbb{R}$,

$$\int_T^{T+H} |\zeta(\sigma + it + i\tau)|^2 dt \ll H(1 + |\tau|).$$

Proof. The estimate of the lemma is obtained in [10] in the proof of Lemma 12. For its proof, Theorem 7.1 of [7] is applied: if (κ, λ) is an exponent pair and σ , $1/2 < \sigma < 1$, is fixed, then for

$$T^{(\kappa+\lambda+1-2\sigma)/2(\kappa+1)} (\log T)^{(2+\kappa)/(\kappa+1)} \leq H \leq T$$

and $1 + \lambda - \kappa \geq 2\sigma$, the estimate

$$\int_{T-H}^{T+H} |\zeta(\sigma + it)|^2 dt \ll H$$

is true uniformly in H . □

To pass from the continuous mean square to a discrete one, we will use the Gallagher lemma.

Lemma 5. *Let $T_0, T \geq \delta > 0$ be real numbers, and \mathcal{T} be a finite set in the interval $[T_0 + \delta/2, T_0 + T - \delta/2]$. Define*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x| < \delta}} 1,$$

and let $S(x)$ be a complex-valued continuous function on $[T_0, T_0 + T]$ having a continuous derivative on $(T_0, T_0 + T)$. Then

$$\begin{aligned} \sum_{t \in \mathcal{T}} N_\delta^{-1}(t) |S(t)|^2 &\leq \frac{1}{\delta} \int_{T_0}^{T_0+T} |S(x)|^2 dx \\ &+ \left(\int_{T_0}^{T_0+T} |S(x)|^2 dx \int_{T_0}^{T_0+T} |S'(x)|^2 dx \right)^{1/2}. \end{aligned}$$

The proof of the lemma can be found in [14, Lemma 1.4].

Lemma 6. *Suppose that $N^{1/3}(\log N)^{26/15} \leq M \leq N$, and $\sigma, 1/2 < \sigma < 1$, and $h > 0$ are fixed. Then, for $\tau \in \mathbb{R}$,*

$$\sum_{k=N}^{N+M} |\zeta(\sigma + ikh + i\tau)|^2 \ll_h M(1 + |\tau|).$$

Proof. In notation of Lemma 5, we take $\delta = h, T_0 = (N - 1/2)h, T = (M + 1)h, \mathcal{T} = \{Nh, (N + 1)h, \dots, (N + M)h\}$ and $S(x) = \zeta(\sigma + ix + i\tau)$. Then

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x| < h}} 1 = 1.$$

Therefore, by Lemma 5,

$$\begin{aligned} \sum_{k=N}^{N+M} |\zeta(s + ikh + i\tau)|^2 &\leq \frac{1}{h} \int_{(N-1/2)h}^{(N+M+1/2)h} |\zeta(\sigma + it + i\tau)|^2 dt \\ (1) \quad &+ \left(\int_{(N-1/2)h}^{(N+M+1/2)h} |\zeta(\sigma + it + i\tau)|^2 dt \int_{(N-1/2)h}^{(N+M+1/2)h} |\zeta'(\sigma + it + i\tau)|^2 dt \right)^{1/2}. \end{aligned}$$

In view of Lemma 4,

$$(2) \quad \frac{1}{h} \int_{(N-1/2)h}^{(N+M+1/2)h} |\zeta(\sigma + it + i\tau)|^2 dt \ll_h M(1 + |\tau|).$$

An application of the Cauchy integral formula together with Lemma 4 shows that

$$\int_{(N-1/2)h}^{(N+M+1/2)h} |\zeta'(\sigma + it + i\tau)|^2 dt \ll_h M(1 + |\tau|).$$

This, (1) and (2) give the estimate of the lemma. □

Now let $\theta > 1/2$ be a fixed number, and, for $m, n \in \mathbb{N}$,

$$v_n(m) = \exp \left\{ - \left(\frac{m}{n} \right)^\theta \right\}$$

Define the function

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s}.$$

Then it is known [8] that the latter series is absolutely convergent for $\sigma > 1/2$. Lemma 6 allows to approximate the function $\zeta(s)$ by $\zeta_n(s)$ in the mean. More precisely, we have the following statement.

Lemma 7. *Suppose that K is a compact subset of the strip D , and $N^{1/3}(\log N)^{26/15} \leq M \leq N$. Then*

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{M} \sum_{k=N}^{N+M} \sup_{s \in K} |\zeta(s + ikh) - \zeta_n(s + ikh)| = 0.$$

Proof. We use the integral representation [8] for the function $\zeta_n(s)$

$$(3) \quad \zeta_n(s) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z) l_n(z) \frac{dz}{z}, \quad \sigma > \frac{1}{2},$$

where

$$l_n(s) = \frac{s}{\theta} \Gamma \left(\frac{s}{\theta} \right) n^s,$$

and $\Gamma(s)$, as usual, denotes the Euler gamma-function. We fix $\varepsilon > 0$ such that $1/2 + 2\varepsilon \leq \sigma \leq 1 - \varepsilon$ for points $s = \sigma + iv \in K$. Let $\hat{\sigma} = \sigma - 1/2 - \varepsilon$. Thus, $\hat{\sigma} > 0$ for points $s \in K$, and, in view of (3),

$$\zeta_n(s) - \zeta(s) = \frac{1}{2\pi i} \int_{-\hat{\sigma}-i\infty}^{-\hat{\sigma}+i\infty} \zeta(s+z) l_n(z) \frac{dz}{z} + \operatorname{Res}_{z=1-s} \frac{l_n(z)}{z} \zeta(s+z).$$

Thus, for $s \in K$,

$$\begin{aligned} \zeta(s + ikh) - \zeta_n(s + ikh) &\ll \int_{-\infty}^{\infty} |\zeta(s + ikh - \hat{\sigma} + it)| \frac{|l_n(-\hat{\sigma} + it)|}{|-\hat{\sigma} + it|} dt \\ &\quad + \frac{|l_n(1-s-ikh)|}{|1-s-ikh|}. \end{aligned}$$

Hence, taking t in place of $t + v$, we find that

$$\begin{aligned} &\zeta(s + ikh) - \zeta_n(s + ikh) \\ &\ll \int_{-\infty}^{\infty} \left| \zeta \left(\frac{1}{2} + \varepsilon + ikh + it \right) \right| \frac{|l_n(1/2 + \varepsilon - s + it)|}{|1/2 + \varepsilon - s + it|} dt + \frac{|l_n(1-s-ikh)|}{|1-s-ikh|}. \end{aligned}$$

This gives

$$(4) \quad \frac{1}{M} \sum_{k=N}^{N+M} \sup_{s \in K} |\zeta(s + ikh) - \zeta_n(s + ikh)| \ll S_1 + S_2,$$

where

$$S_1 = \int_{-\infty}^{\infty} \left(\frac{1}{M} \sum_{k=N}^{N+M} \left| \zeta \left(\frac{1}{2} + \varepsilon + ikh + it \right) \right| \right) \sup_{s \in K} \frac{|l_n(1/2 + \varepsilon - s + it)|}{|1/2 + \varepsilon - s + it|} dt$$

and

$$S_2 = \frac{1}{M} \sum_{k=N}^{N+M} \sup_{s \in K} \frac{|l_n(1 - s - ikh)|}{|1 - s - ikh|}.$$

We use the estimate

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad |t| \geq t_0,$$

which is uniform for $\sigma_1 \leq \sigma \leq \sigma_2$ with arbitrary $\sigma_1 < \sigma_2$. The latter estimate together with the definition of $l_n(s)$ gives

$$\frac{l_n(1/2 + \varepsilon - \sigma - iv + it)}{1/2 + \varepsilon - \sigma - iv + it} \ll \frac{n^{1/2+\varepsilon-\sigma}}{\theta} \exp\left\{-\frac{c}{\theta}|t-v|\right\} \ll_{\theta,K} n^{-\varepsilon} \exp\left\{-\frac{c}{\theta}|t|\right\}.$$

This and Lemma 6 show that

$$(5) \quad S_1 \ll_{h,\theta,K} n^{-\varepsilon} \int_{-\infty}^{\infty} (1 + |t|) \exp\left\{-\frac{c}{\theta}|t|\right\} dt \ll_{h,\theta,K} n^{-\varepsilon}.$$

Similarly, we find that, for $s \in K$,

$$\frac{l_n(1 - s - ikh)}{1 - s - ikh} \ll_{\theta,K} n^{1-\sigma} \exp\left\{-\frac{c}{\theta}kh\right\}.$$

Therefore,

$$S_2 \ll_{\theta,K} n^{1/2-2\varepsilon} \frac{1}{M} \sum_{k=N}^{N+M} \exp\left\{-\frac{c}{\theta}kh\right\} \ll_{h,\theta,K} \frac{n^{1/2-2\varepsilon}}{M}.$$

This, (4) and (5) imply the estimate

$$\frac{1}{M} \sum_{k=N}^{N+M} \sup_{s \in K} |\zeta(s + ikh) - \zeta_n(s + ikh)| \ll_{h,\theta,K} \left(n^{-\varepsilon} + \frac{n^{1/2-2\varepsilon}}{M} \right).$$

Now, letting $N \rightarrow \infty$ (then $M \rightarrow \infty$), and then $n \rightarrow \infty$, we obtain the equality of the lemma. \square

3. LIMIT THEOREMS

Denote by $\mathcal{B}(\mathbb{X})$ be the Borel σ -field of the space \mathbb{X} . In this section, we will consider the weak convergence of

$$P_{N,M,h}(A) \stackrel{def}{=} \frac{1}{M+1} \# \{N \leq k \leq N+M : \zeta(s+ikh) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

as $N \rightarrow \infty$. For the statement of a limit theorem, we will use the following notation. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, \mathbb{P} be the set of all prime numbers, and

$$\Omega = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H exists, and we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the p th component of an element $\omega \in \Omega$, and on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$ -valued random element $\zeta(s, \omega)$ by the formula

$$\zeta(s, \omega) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\omega(p)}{p^s} \right)^{-1}.$$

Let P_ζ be the distribution of $\zeta(s, \omega)$, that is,

$$P_\zeta(A) = m_H \{ \omega \in \Omega : \zeta(s, \omega) \in A \}, \quad A \in \mathcal{B}(H(D)).$$

Let

$$L(\mathbb{P}, h, \pi) = \left\{ (\log p : p \in \mathbb{P}), \frac{2\pi}{h} \right\}.$$

The set $L(\mathbb{P}, h, \pi)$ consists of logarithms of all prime numbers and the number $2\pi/h$. We note that, in general,

$$(6) \quad L(\mathbb{P}, h, \pi) \neq \{(\log p : p \in \mathbb{P})\} \cup \left\{ \frac{2\pi}{h} \right\}$$

because if $2\pi/h = \log p_0$ for some $p_0 \in \mathbb{P}$, then the right-hand side of (6) is $\{\log p : p \in \mathbb{P}\}$. However, in this case the $L(\mathbb{P}, h, \pi)$ is linearly dependent as having two the same elements.

Theorem 8. *Suppose that the set $L(\mathbb{P}, h, \pi)$ is linearly independent over \mathbb{Q} . Then $P_{N,M,h}$ converges weakly to the measure P_ζ as $N \rightarrow \infty$.*

The case of the linear dependence of the set $L(\mathbb{P}, h, \pi)$ is more complicated, and we need some additional arguments.

So, suppose that the set $L(\mathbb{P}, h, \pi)$ is linearly dependent over \mathbb{Q} . Then, clearly, there exist $m \in \mathbb{N}$ such that $\exp\{(2\pi m)/h\}$ is a rational number. Let m_0 be the smallest of them, and suppose that

$$\exp\left\{\frac{2\pi m_0}{h}\right\} = \frac{a}{b}$$

with $a, b \in \mathbb{N}$, $(a, b) = 1$. Let \mathbb{P}_0 be a finite subset of \mathbb{P} defined by

$$\mathbb{P}_0 = \left\{ p \in \mathbb{P} : \alpha_p \neq 0 \text{ in } \frac{a}{b} = \prod_{p \in \mathbb{P}} p^{\alpha_p} \right\}.$$

Denote by Ω_h the closed subgroup of the group Ω generated by $\{p^{-ih} : p \in \mathbb{P}\}$. Then, on $(\Omega_h, \mathcal{B}(\Omega_h))$, the probability Haar measure m_H^h exists, and we have the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$. By the Bagchi lemma [1], see, also [11, Lemma 1],

$$\Omega_h = \{\omega \in \Omega : \omega(a) = \omega(b)\}$$

if the set $L(\mathbb{P}, h, \pi)$ is linearly dependent over \mathbb{Q} .

On $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$, define the $H(D)$ -valued random element

$$\zeta_h(s, \omega) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

Then $\zeta_h(s, \omega)$ is the restriction of $\zeta(s, \omega)$ to the space $(\Omega_h, \mathcal{B}(\Omega_h))$. Denote by $P_{\zeta, h}$ the distribution of $\zeta_h(s, \omega)$, that is,

$$P_{\zeta, h}(A) = m_H^h\{\omega \in \Omega_h : \zeta_h(s, \omega) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

Theorem 9. *Suppose that the set $L(\mathbb{P}, h, \pi)$ is linearly dependent over \mathbb{Q} . Then $P_{N, M, h}$ converges weakly to the measure $P_{\zeta, h}$ as $N \rightarrow \infty$.*

We start the proofs of Theorems 8 and 9 with limit theorems on Ω and Ω_h , respectively.

For $A \in \mathcal{B}(\Omega)$, define

$$Q_{N, M}(A) = \frac{1}{M+1} \#\{N \leq k \leq N+M : (p^{-ikh} : p \in \mathbb{P}) \in A\}.$$

Lemma 10. *Suppose that the set $L(\mathbb{P}, h, \pi)$ is linearly independent over \mathbb{Q} . Then $Q_{N, M}$ converges weakly to the Haar measure m_H as $N \rightarrow \infty$.*

Proof. Let $g_{N, M}(\underline{k})$, $\underline{k} = (k_p : k_p \in \mathbb{Z}, p \in \mathbb{P})$, be the Fourier transform of $Q_{N, M}$, i.e.,

$$\begin{aligned} g_{N, M}(\underline{k}) &= \int_{\Omega} \prod'_{p \in \mathbb{P}} \omega^{k_p}(p) \, dQ_{N, M} = \frac{1}{M+1} \sum_{k=N}^{N+M} \prod'_{p \in \mathbb{P}} p^{-ik_p kh} \\ (7) \quad &= \frac{1}{M+1} \sum_{k=N}^{N+M} \exp\left\{-ikh \sum'_{p \in \mathbb{P}} k_p \log p\right\}, \end{aligned}$$

where the sign “'” means that only a finite number of integers k_p are distinct from zero. Obviously,

$$(8) \quad g_{N,M}(\underline{0}) = 1.$$

Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers, we have that

$$\sum'_{p \in \mathbb{P}} k_p \log p \neq 0$$

for $\underline{k} \neq \underline{0}$. Thus, in this case

$$\exp \left\{ -ih \sum'_{p \in \mathbb{P}} k_p \log p \right\} \neq 1.$$

Actually, if the latter inequality is not true, then

$$\sum'_{p \in \mathbb{P}} k_p \log p = \frac{2\pi r}{h}$$

with some $r \in \mathbb{Z}$, and this contradicts the linear independence over \mathbb{Q} of the set $L(\mathbb{P}, h, \pi)$. Therefore, for $\underline{k} \neq \underline{0}$, (7) implies that

$$g_{N,M}(\underline{k}) = \frac{\exp \left\{ -iNh \sum'_{p \in \mathbb{P}} k_p \log p \right\} - \exp \left\{ -i(N+M+1)h \sum'_{p \in \mathbb{P}} k_p \log p \right\}}{M \left(1 - \exp \left\{ -ih \sum'_{p \in \mathbb{P}} k_p \log p \right\} \right)}.$$

This and (8) show that

$$\lim_{N \rightarrow \infty} g_{N,M}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}, \end{cases}$$

and the lemma follows by a continuity theorem for probability measures on compact groups. \square

For $A \in \mathcal{B}(\Omega_h)$, define

$$Q_{N,M,h}(A) = \frac{1}{M+1} \# \{ N \leq k \leq N+M : (p^{-ikh} : p \in \mathbb{P}) \in A \}.$$

Lemma 11. *Suppose that the set $L(\mathbb{P}, h, \pi)$ is linearly dependent over \mathbb{Q} . Then $Q_{N,M,h}$ converges weakly to the Haar measure m_H^h as $N \rightarrow \infty$.*

Proof. Denote by \mathcal{D} the dual group of Ω . In the proof Lemma 10, we already have used that the characters $\chi \in \mathcal{D}$ are of the form

$$\chi(\omega) = \prod'_{p \in \mathbb{P}} \omega^{k_p}(p).$$

Let the character $\chi_{m_0}(\omega) \in \mathcal{D}$ be given by

$$(9) \quad \chi_{m_0}(\omega) = \prod_{p \in \mathbb{P}_0} \omega^{\alpha_p}(p) = \frac{\omega(a)}{\omega(b)},$$

and $\Omega_h^\perp = \{\chi \in \mathcal{D} : \chi(\omega) = 1, \omega \in \Omega_h\}$. Then, by (2.2) of [11],

$$\Omega_h^\perp = \{\chi_{m_0}^l : l \in \mathbb{Z}\}.$$

Thus, in view of Theorem 27 from [15], $\mathcal{D}_h = \mathcal{D}/\Omega_h^\perp$ is the dual group of Ω_h . Hence, the characters χ of the group Ω_h are of the form

$$(10) \quad \chi(\omega) = \prod'_{p \in \mathbb{P} \setminus \mathbb{P}_0} \omega^{k_p}(p) \prod_{p \in \mathbb{P}_0} \omega^{k_p + l\alpha_p}(p), \quad l \in \mathbb{Z}.$$

Denote by $g_{N,M,h}(\underline{k})$, $\underline{k} = (k_p : k_p \in \mathbb{Z}, p \in \mathbb{P})$, the Fourier transform of $Q_{N,M,h}$. Then, in virtue of (10),

$$(11) \quad \begin{aligned} g_{N,M,h}(\underline{k}) &= \int_{\Omega_h} \chi(\omega) \, dQ_{N,M,h} \\ &= \frac{1}{M+1} \sum_{k=N}^{N+M} \prod'_{p \in \mathbb{P} \setminus \mathbb{P}_0} p^{-ik_p kh} \prod_{p \in \mathbb{P}_0} p^{-ikh(k_p + l\alpha_p)}, \quad l \in \mathbb{Z}. \end{aligned}$$

Consider two cases 1° and 2°.

1°. Suppose that $k_p = 0$ for any $p \in \mathbb{P} \setminus \mathbb{P}_0$ and $k_p = r\alpha_p$ for any $p \in \mathbb{P}_0$ with some $r \in \mathbb{Z}$. Then, taking into account (9), we obtain from (11) that

$$(12) \quad g_{N,M,h}(\underline{k}) = 1.$$

2°. Suppose that either $k_p \neq 0$ for some $p \in \mathbb{P} \setminus \mathbb{P}_0$, or there does not exist $r \in \mathbb{Z}$ such that $k_p = r\alpha_p$ for all $p \in \mathbb{P}_0$. We observe that in this case

$$(13) \quad A(h) \stackrel{def}{=} \exp \left\{ -ih \left(\sum'_{p \in \mathbb{P} \setminus \mathbb{P}_0} k_p \log p + \sum_{p \in \mathbb{P}_0} (k_p + l\alpha_p) \log p \right) \right\} \neq 1.$$

Actually, if (13) is not true, then

$$(14) \quad \exp \left\{ \sum'_{p \in \mathbb{P} \setminus \mathbb{P}_0} k_p \log p + \sum_{p \in \mathbb{P}_0} (k_p + l\alpha_p) \log p \right\} = \frac{2\pi v}{h}$$

with some $v \in \mathbb{Z}$. If v is a multiple of m_0 , then

$$\exp \left\{ \frac{2\pi v}{h} \right\} = \prod_{p \in \mathbb{P}_0} p^{v_1 \alpha_p}, \quad v_1 \in \mathbb{Z},$$

hence,

$$\sum'_{p \in \mathbb{P} \setminus \mathbb{P}_0} k_p \log p + \sum_{p \in \mathbb{P}_0} (k_p + v_2 \alpha_p) \log p = 0$$

with some $v_2 \in \mathbb{Z}$. Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over \mathbb{Q} , the latter equality leads to contradiction. Now, suppose that v is not a multiple of m_0 . Then the number $\exp\{(2\pi v)/h\}$ is irrational, and we again have a contradiction because the left-hand side of (14) is a rational number. Thus, inequality (13) is true, and we find from (11) that

$$\begin{aligned} g_{N,M,h}(\underline{k}) &= \frac{1}{M+1} \sum_{k=N}^{N+M} \exp \left\{ -ikh \left(\sum'_{p \in \mathbb{P} \setminus \mathbb{P}_0} k_p \log p + \sum_{p \in \mathbb{P}_0} (k_p + l \alpha_p) \log p \right) \right\} \\ &= \frac{A^N(h) - A^{N+M+1}(h)}{M(1 - A(h))}. \end{aligned}$$

This and (12) show that

$$\lim_{N \rightarrow \infty} g_{N,M,h}(\underline{k}) = \begin{cases} 1 & \text{for case } 1^\circ, \\ 0 & \text{for case } 2^\circ. \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H^h , the lemma is a consequence of a continuity theorem for probability measures on compact groups. \square

For $A \in \mathcal{B}(H(D))$, define

$$P_{N,M,n,h}(A) = \frac{1}{M+1} \# \{N \leq k \leq N+M : \zeta_n(s + ikh) \in A\}.$$

Consider the function $u_n : \Omega \rightarrow H(D)$ given by

$$u_n(\omega) = \sum_{m=1}^{\infty} \frac{\omega(m)v_n(m)}{m^s}, \quad \omega \in \Omega,$$

where, for $m \in \mathbb{N}$,

$$\omega(m) = \prod_{\substack{p^\alpha | m \\ p^{\alpha+1} \nmid m}} \omega^\alpha(p).$$

Then the series for $u_n(\omega)$ also converges absolutely for $\sigma > 1/2$, hence, the function u_n is continuous, thus, $(\mathcal{B}(\Omega), \mathcal{B}(H(D)))$ -measurable. Therefore, the Haar measure on $(\Omega, \mathcal{B}(\Omega))$ induces the unique probability measure $V_n \stackrel{def}{=} m_H u_n^{-1}$, where, for $A \in \mathcal{B}(H(D))$,

$$m_H u_n^{-1}(A) = m_H(u_n^{-1}A).$$

Lemma 12. *Suppose that the set $L(\mathbb{P}, h, \pi)$ is linearly independent over \mathbb{Q} . Then $P_{N,M,n,h}$ converges weakly to the measure V_n as $N \rightarrow \infty$.*

Proof. The lemma follows from Lemma 10, equality $P_{N,M,n,h} = Q_{N,M,h}u_n^{-1}$, continuity of the function u_n and Theorem 5.1 of [2]. \square

Now, let the function $u_{n,h} : \Omega_h \rightarrow H(D)$ be defined by

$$u_{n,h}(\omega) = \sum_{m=1}^{\infty} \frac{\omega(m)v_n(m)}{m^s}, \quad \omega \in \Omega_h,$$

and $V_{n,h} = m_H^h u_{n,h}^{-1}$. Using Lemma 11 in place of Lemma 10 and repeating the proof of Lemma 12, we arrive to the following statement.

Lemma 13. *Suppose that the set $L(\mathbb{P}, h, \pi)$ is linearly dependent over \mathbb{Q} . Then $P_{N,M,n,h}$ converges weakly to the measure $V_{n,h}$ as $N \rightarrow \infty$.*

Proof of Theorem 8. First, we will prove that the sequence $\{V_n : n \in \mathbb{N}\}$ is relatively compact. In view of the Prokhorov theorem [2, Theorem 6.1], it is sufficient to show that the sequence $\{V_n : n \in \mathbb{N}\}$ is tight. Let $\xi_{N,M}$ be a discrete random variable defined on a certain probability space with a measure μ such that

$$\mu\{\xi_{N,M} = kh\} = \frac{1}{M+1}, \quad k = N, \dots, N+M.$$

Denote by X_n the $H(D)$ -valued random element with distribution V_n , and define one more $H(D)$ -valued random element

$$X_{N,M,n} = X_{N,M,n}(s) = \zeta_n(s + i\xi_{N,M}).$$

Then, by Lemma 12, we have the relation

$$(15) \quad X_{N,M,n} \xrightarrow[N \rightarrow \infty]{\mathcal{D}} X_n.$$

Let $\{K_l : l \in \mathbb{N}\} \subset D$ be a sequence of compact sets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some l . The function $\rho : H(D) \times H(D) \rightarrow \mathbb{R}$ defined by

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

is a metric in $H(D)$ inducing its topology of uniform convergence on compacta. Using Lemma 6, the Cauchy integral formula and Lemma 7, we find that

$$(16) \quad \sup_{n \in \mathbb{N}} \limsup_{N \rightarrow \infty} \frac{1}{M+1} \sum_{K=N}^{N+M} \sup_{s \in K_l} |\zeta_n(s + ikh)| \ll R_l < \infty.$$

We fix $\varepsilon > 0$ and put $\hat{M} = \hat{M}_l(\varepsilon) = 2^l R_l \varepsilon^{-1}$. Then, in virtue of (16),

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \mu \left\{ \sup_{s \in K_l} |X_{N,M,n}(s)| > \hat{M} \right\} \\ &= \limsup_{N \rightarrow \infty} \frac{1}{M+1} \# \left\{ N \leq k \leq N+M : \sup_{s \in K_l} |\zeta_n(s+ikh)| > \hat{M} \right\} \\ &\leq \sup_{n \in \mathbb{N}} \limsup_{N \rightarrow \infty} \frac{1}{(M+1)\hat{M}} \sum_{k=N}^{N+M} \sup_{s \in K_l} |\zeta_n(s+ikh)| \leq \frac{\varepsilon}{2^l}. \end{aligned}$$

This and (15) imply

$$(17) \quad \mu \left\{ \sup_{s \in K_l} |X_n| > \hat{M} \right\} \leq \frac{\varepsilon}{2^l}$$

for all $n \in \mathbb{N}$ and $l \in \mathbb{N}$. The set

$$K = K(\varepsilon) = \left\{ g \in H(D) : \sup_{s \in K_l} |g(s)| \leq \hat{M}_l(\varepsilon), l \in \mathbb{N} \right\}$$

is compact in the space $H(D)$, and, by (17),

$$\mu(X_n \in K) \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Thus,

$$V_n(K) \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}$, i.e., the sequence $\{V_n\}$ is tight.

Since the sequence $\{V_n : n \in \mathbb{N}\}$ is relatively compact, there exists a subsequence $\{V_{n_r}\} \subset \{V_n\}$ such that V_{n_r} converges weakly to a certain probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $r \rightarrow \infty$, i.e.,

$$(18) \quad X_{n_r} \xrightarrow[r \rightarrow \infty]{\mathcal{D}} P.$$

Introduce one more $H(D)$ -valued random element

$$Y_{N,M} = Y_{N,M}(s) = \zeta(s + i\xi_{N,M}).$$

Then, by Lemma 7 and the definition of the metric ρ , we obtain that, for every $\varepsilon > 0$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \mu \{ \rho(Y_{N,M}, X_{N,M,n}) \geq \varepsilon \} \\ &= \lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{M+1} \# \{ N \leq k \leq N+M : \rho(\zeta(s+ikh), \zeta_n(s+ikh)) \geq \varepsilon \} \\ &\leq \lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{(M+1)\varepsilon} \sum_{k=N}^{N+M} \rho(\zeta(s+ikh), \zeta_n(s+ikh)) = 0. \end{aligned}$$

This, (15), (18) and Theorem 4.2 of [2] show that

$$(19) \quad Y_{N,M} \xrightarrow[N \rightarrow \infty]{\mathcal{D}} P,$$

and this means that $P_{N,M,h}$ converges weakly to P as $N \rightarrow \infty$.

For identification of the limit measure P , we observe that relation (19) shows that the measure P is independent of the choice of the sequence $\{V_{n_r}\}$. Therefore, we have that

$$X_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} P,$$

thus, $P_{N,M,h}$, as $N \rightarrow \infty$, converges weakly to the limit measure P of V_n as $n \rightarrow \infty$. It is well known, see, for example, [8], [3], that

$$\frac{1}{T} \text{meas}\{\tau \in [0, T] : \zeta(s + i\tau) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

also, as $T \rightarrow \infty$, converges weakly to the limit measure P of V_n , and that P coincides with P_ζ . Thus, $P_{N,M,h}$ also converges weakly to P_ζ as $N \rightarrow \infty$. The theorem is proved. \square

Proof of Theorem 9 is more complicated than that of Theorem 8 because we have not an analogue of a limit theorem for the measure $V_{n,h}$. Therefore, we start the proof of Theorem 9 with the following lemma.

Lemma 14. *Suppose that the set $L(\mathbb{P}, h, \pi)$ is linearly dependent over \mathbb{Q} . Then $P_{N,M,h}$ converges weakly to the limit measure V_h of $V_{n,h}$ as $n \rightarrow \infty$.*

Proof. We repeat the proof of Theorem 8, and in place of Lemma 12, apply Lemma 13. \square

We can't give a direct identification of the measure V_h in Lemma 14 because the validity of the Birkhoff-Khinchine ergodic theorem in short interval is not known. Therefore, first we will give a sketch of the proof of a limit theorem for

$$P_{N,h}(A) = \frac{1}{N+1} \#\{0 \leq k \leq N : \zeta_n(s + ikh) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

as $N \rightarrow \infty$, when the set $L(\mathbb{P}, h, \pi)$ is linearly dependent over \mathbb{Q} .

For $\omega \in \Omega_h$, define the transformation $\varphi_h(\omega)$ by the formula

$$\varphi_h(\omega) = (p^{-ih} : p \in \mathbb{P}) \omega.$$

Then φ_h is a measurable measure-preserving transformation on the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$. A set $A \in \mathcal{B}(\Omega_h)$ is called invariant with respect to the transformation φ_h if the sets A and $\varphi_h(A)$ can differ from each other at most by a set of m_H^h -measure zero. The transformation φ_h is called ergodic if the σ -field of invariant sets consists only of the sets of m_H^h -measure zero or one.

Lemma 15. *The transformation φ_h is ergodic.*

Proof. The lemma is Lemma 3 of [11] because the linear dependence of the set $L(\mathbb{P}, h, \pi)$ implies that h is of type 2. \square

For convenience, we remind of the classical Birkhoff-Khinchine ergodic theorem.

Lemma 16. *Let φ be a measure preserving ergodic transformation on a certain probability space with measure μ , and $\xi = \xi(\omega)$ be a random variable, $\mathbb{E}|\xi| < \infty$. Then μ -almost surely*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \xi(\varphi^k(\omega)) = \mathbb{E}\xi.$$

Proof of the lemma can be found, for example, in [19], Theorem V.3.1.

Lemma 16 allows to estimate the discrete mean square of the function $\zeta(s, \omega)$.

Lemma 17. *Suppose that $\sigma, 1/2 < \sigma < 1$, is fixed and $\tau \in \mathbb{R}$. Then, for almost all $\omega \in \Omega_h$,*

$$\sum_{k=0}^N |\zeta(\sigma + ikh + i\tau, \omega)|^2 \ll N(1 + |\tau|).$$

Proof. Suppose that $|\hat{\tau}| < h$. By the definition of φ_h ,

$$|\zeta(\sigma + ikh + i\hat{\tau}, \omega)|^2 = |\zeta(\sigma + i\hat{\tau}, \varphi_h^k(\omega))|^2.$$

Therefore, by Lemmas 15 and 16, and equalities

$$\omega(m)\omega(a^k b^l)\overline{\omega}(m)\overline{\omega}(b^k a^l) = 1, \quad m \in \mathbb{N}, k, l \in \mathbb{N}_0,$$

we find that

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N |\zeta(\sigma + ikh + i\hat{\tau}, \omega)|^2 &= \mathbb{E} \left| \sum_{m=1}^{\infty} \frac{\omega(m)}{m^{\sigma+i\hat{\tau}}} \right|^2 \\ &= \sum_{m=1}^{\infty} \frac{1}{m^{2\sigma}} + \sum_{m=1}^{\infty} \frac{1}{m^{2\sigma}} \sum_{\substack{k=1 \\ k \neq l}}^{\infty} \sum_{l=1}^{\infty} \frac{1}{a^{(\sigma+i\hat{\tau})k} b^{(\sigma+i\hat{\tau})l}} \frac{1}{a^{(\sigma-i\hat{\tau})l} b^{(\sigma-i\hat{\tau})k}} \\ &= \sum_{m=1}^{\infty} \frac{1}{m^{2\sigma}} \left(1 + \frac{\frac{1}{a^{\sigma+i\hat{\tau}} b^{\sigma-i\hat{\tau}}}}{1 - \frac{1}{a^{\sigma+i\hat{\tau}} b^{\sigma-i\hat{\tau}}}} \frac{\frac{1}{a^{\sigma-i\hat{\tau}} b^{\sigma+i\hat{\tau}}}}{1 - \frac{1}{a^{\sigma-i\hat{\tau}} b^{\sigma+i\hat{\tau}}}} - \frac{\frac{1}{(ab)^{2\sigma}}}{1 - \frac{1}{(ab)^{2\sigma}}} \right) \\ &\leq \sum_{m=1}^{\infty} \frac{1}{m^{2\sigma}} \left(1 + \frac{1}{|a^{\sigma+i\hat{\tau}} b^{\sigma-i\hat{\tau}} - 1|^2} + \frac{1}{(ab)^{2\sigma} - 1} \right) < \infty. \end{aligned}$$

Thus,

$$\sum_{k=0}^N |\zeta(\sigma + ikh + i\hat{\tau}, \omega)|^2 \ll N.$$

Hence, for all $\tau \in \mathbb{R}$,

$$\begin{aligned} \sum_{k=0}^N |\zeta(\sigma + ikh + i\tau, \omega)|^2 &= \sum_{k=0}^N \left| \zeta\left(\sigma + ikh + \left[\frac{\tau}{h}\right]h + \hat{\tau}, \omega\right) \right|^2 \\ &= \sum_{k=\lceil \tau/h \rceil}^{N+\lceil \tau/h \rceil} |\zeta(\sigma + ikh + i\hat{\tau}, \omega)|^2 \ll N(1 + |\tau|). \end{aligned}$$

□

Lemma 18. *The measure*

$$Q_{N,h}(A) = \frac{1}{N+1} \#\{0 \leq k \leq N : (p^{-ikh} : p \in \mathbb{P}) \in A\}, \quad A \in \mathcal{B}(\Omega_h),$$

converges weakly to the Haar measure m_H^h as $N \rightarrow \infty$.

Proof. The lemma is Lemma 2 of [11].

□

For $A \in \mathcal{B}(H(D))$ and $\omega \in \Omega_h$, define

$$P_{N,n,h}(A) = \frac{1}{N+1} \#\{0 \leq k \leq N : \zeta_n(s + ikh) \in A\}$$

and

$$P_{N,n,h,\omega}(A) = \frac{1}{N+1} \#\{0 \leq k \leq N : \zeta_n(s + ikh, \omega) \in A\}.$$

Lemma 19. *The measures $P_{N,n,h}$ and $P_{N,n,h,\omega}$ both converge weakly to the same probability measure $V_{n,h}$ on $(H(D), \mathcal{B}(H(D)))$ as $N \rightarrow \infty$.*

Proof. We use Lemma 18 and repeat the proof of Lemmas 12 and 13, and apply the invariance of the Haar measure m_H^h . □

Lemma 20. *Suppose that $K \subset D$ is a compact set. Then*

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \sup_{s \in K} |\zeta(s + ikh) - \zeta_n(s + ikh)| = 0$$

and, for almost all $\omega \in \Omega_h$,

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \sup_{s \in K} |\zeta(s + ikh, \omega) - \zeta_n(s + ikh, \omega)| = 0.$$

Proof. It is well known that, for fixed σ , $1/2 < \sigma < 1$,

$$\int_0^T |\zeta(\sigma + it)|^2 dt \ll T.$$

From this and Lemma 5, we deduce that, for $1/2 < \sigma < 1$ and $\tau \in \mathbb{R}$,

$$\sum_{k=0}^N |\zeta(\sigma + i\tau + ikh)|^2 \ll_h N(1 + |\tau|).$$

Therefore, reasoning as in the proof of Lemma 7, leads to the first equality of the lemma. The second equality of the lemma is obtained similarly by using Lemma 17. \square

For $A \in \mathcal{B}(H(D))$ and $\omega \in \Omega_h$, define

$$P_{N,h,\omega}(A) = \frac{1}{N+1} \#\{0 \leq k \leq N : \zeta(s + ikh, \omega) \in A\}$$

Lemma 21. *The measures $P_{N,h}$ and $P_{N,h,\omega}$ as $N \rightarrow \infty$ both converge weakly to the limit measure P_h of $V_{n,h}$ as $n \rightarrow \infty$.*

Proof. We use Lemmas 19 and 20 and follow the proof of Theorem 8. \square

We will use two following equivalents of weak convergence of probability measures.

Lemma 22. *Let P_n , $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then the statements*

- 1° P_n converges weakly to P as $n \rightarrow \infty$;
- 2° For every open set $G \subset \mathbb{X}$,

$$\liminf_{n \rightarrow \infty} P_n(G) \geq P(G);$$

- 3° For every continuity set A of the measure P ,

$$\lim_{n \rightarrow \infty} P_n(A) = P(A)$$

are equivalent.

The lemma is a part of Theorem 2.1 of [2].

Lemma 23. *The measure P_h in Lemma 21 coincides with $P_{\zeta,h}$.*

Proof. Let $A \in \mathcal{B}(H(D))$ be a continuity set of the measure P_h ($P_h(\partial A) = 0$, where ∂A is a boundary of A). Then, in view of Lemma 21 and 3° of Lemma 22,

$$(20) \quad \lim_{N \rightarrow \infty} P_{N,h,\omega}(A) = P_h(A).$$

On the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$, define the random variable η by

$$\eta(\omega) = \begin{cases} 1 & \text{if } \zeta_h(s, \omega) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of η , we have

$$(21) \quad \mathbb{E}\eta = \int_{\Omega_h} \eta \, d m_H^h = m_H^h \{ \omega \in \Omega_h : \zeta_h(s, \omega) \in A \} = P_{\zeta, h}(A).$$

In virtue of Lemmas 15 and 16, we have

$$(22) \quad \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \eta(\varphi^k(\omega)) = \mathbb{E}\eta$$

for almost all $\omega \in \Omega_h$. However, by the definitions of η and φ_h ,

$$\frac{1}{N+1} \sum_{k=0}^N \eta(\varphi^k(\omega)) = \frac{1}{N+1} \#\{0 \leq k \leq N : \zeta(s + ikh, \omega) \in A\}.$$

From this, and (21) and (22), we find that

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \#\{0 \leq k \leq N : \zeta(s + ikh, \omega) \in A\} = P_{\zeta, h}(A).$$

This and (20) show that $P_h(A) = P_{\zeta, h}(A)$ for every continuity set A of the measure P_h . Since all continuity sets constitute the determining class [2], we have that $P_h = P_{\zeta, h}$. \square

Proof of Theorem 9. By Lemma 14, $P_{N, M, h}$ converges weakly to the measure V_h as $N \rightarrow \infty$. However, by Lemma 23, the measure V_h coincides with $P_{\zeta, h}$. \square

It is convenient to connect Theorems 8 and 9. Let

$$\hat{P}_{\zeta, h} = \begin{cases} P_{\zeta} & \text{if } L(\mathbb{P}, h, \pi) \text{ is linearly independent,} \\ P_{\zeta, h} & \text{if } L(\mathbb{P}, h, \pi) \text{ is linearly dependent.} \end{cases}$$

Then we have the following corollary from Theorems 8 and 9.

Corollary 24. $P_{N, M, h}$ converges weakly to $\hat{P}_{\zeta, h}$ as $N \rightarrow \infty$.

For $F : H(D) \rightarrow H(D)$, define

$$P_{N, M, F, h}(A) = \frac{1}{M+1} \#\{N \leq k \leq N+M : F(\zeta(s + ikh)) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

Theorem 25. Suppose that $F : H(D) \rightarrow H(D)$ is a continuous operator. Then $P_{N, M, F, h}$ converges weakly to the measure $\hat{P}_{\zeta, h} F^{-1}$ as $N \rightarrow \infty$.

Proof. Since $P_{N, M, F, h} = P_{N, M, h} F^{-1}$, the theorem is corollary of Corollary 24, continuity of F and Theorem 5.1 of [2]. \square

4. PROOF OF UNIVERSALITY

We recall the Mergelyan theorem on the approximation of analytic functions by polynomials [13].

Lemma 26. *Suppose that $K \subset \mathbb{C}$ is a compact set with connected complement, and the function $f(s)$ is continuous on K and analytic in the interior of K . Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s \in K} |f(s) - p(s)| < \varepsilon.$$

For proving universality theorems, we additionally need the explicit form of the support of the measure $\hat{P}_{\zeta, h}$. We recall that the support of $\hat{P}_{\zeta, h}$ is a minimal closed set $S_{\zeta, h} \subset H(D)$, such that $\hat{P}_{\zeta, h}(S_{\zeta, h}) = 1$. The set $S_{\zeta, h}$ consists of all elements $g \in H(D)$ such that, for every open neighbourhood G of g , the inequality $\hat{P}_{\zeta, h}(G) > 0$ holds. We remind that

$$S = \{g(s) \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$$

Lemma 27. *The support of $\hat{P}_{\zeta, h}$ is the set S .*

Proof. It is well known, see, for example, [8], that the support of P_{ζ} is the set S . Thus, it remains to consider the case of the measure $P_{\zeta, h}$.

We write the random element $\zeta_h(s, \omega)$ in the form

$$\zeta_h(s, \omega) = \prod_{p \in \mathbb{P}_0} \left(1 - \frac{\omega(p)}{p^s}\right)^{-1} \prod_{p \in \mathbb{P} \setminus \mathbb{P}_0} \left(1 - \frac{\omega(p)}{p^s}\right)^{-1} \stackrel{\text{def}}{=} XY.$$

For $p \in \mathbb{P} \setminus \mathbb{P}_0$, the random variables $\omega(p)$ are independent. Therefore, repeating the proof of Lemma 6.5.5 of [8] that proves that the support of P_{ζ} is the set S , we obtain that the support of random element Y is the set S . The random elements X and Y are independent, moreover, X is not degenerate at zero. Therefore, the support of $\zeta_h(s, \omega)$ is the set S . Since the support of $\zeta_h(s, \omega)$ is the support of the measure $P_{\zeta, h}$, the lemma is proved. \square

Proof of Theorem 1. First part. By Lemma 26, there exists a polynomial $p(s)$ such that

$$(23) \quad \sup_{s \in K} |f(s) - e^{p(s)}| < \frac{\varepsilon}{2}.$$

Since, in view of Lemma 26, the function $e^{p(s)}$ is an element of the support of the measure $\hat{P}_{\zeta, h}$, we have that

$$(24) \quad \hat{P}_{\zeta, h}(G_{\varepsilon}) > 0,$$

where

$$G_{\varepsilon} = \left\{g \in H(D) : \sup_{s \in K} |g(s) - e^{p(s)}| < \frac{\varepsilon}{2}\right\}.$$

This, the definitions of $P_{N,M,h}$ and G_ε , Corollary 24 and 2° of Lemma 22 give the inequality

$$\liminf_{N \rightarrow \infty} \frac{1}{M+1} \# \left\{ N \leq k \leq N+M : \sup_{s \in K} \left| \zeta(s+ikh) - e^{p(s)} \right| < \frac{\varepsilon}{2} \right\} > 0.$$

Combining this with (23) gives the first inequality of the theorem.

Second part. Define the set

$$\hat{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then the boundary $\partial \hat{G}_\varepsilon$ lies in the set

$$\left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \right\},$$

therefore, $\partial \hat{G}_{\varepsilon_1} \cap \partial \hat{G}_{\varepsilon_2} = \emptyset$ for different positive ε_1 and ε_2 . Hence, $\hat{P}_{\zeta,h}(\hat{G}_\varepsilon) > 0$ for at most countably many $\varepsilon > 0$. In other words, the set \hat{G}_ε is a continuity set of the measure $\hat{P}_{\zeta,h}$ for all but at most countably many $\varepsilon > 0$. Thus, Corollary 24 and 3° of Lemma 22 imply the equality

$$(25) \quad \lim_{N \rightarrow \infty} P_{N,M,h}(\hat{G}_\varepsilon) = \hat{P}_{\zeta,h}(\hat{G}_\varepsilon)$$

for all but at most countably many $\varepsilon > 0$. However, inequality (23) shows that $G_\varepsilon \subset \hat{G}_\varepsilon$. Thus $\hat{P}_{\zeta,h}(\hat{G}_\varepsilon) \geq \hat{P}_{\zeta,h}(G_\varepsilon)$, and the second assertion of the theorem follows by (24) and (25). \square

Proof of Theorem 2. First, we will find the support of the measure $\hat{P}_{\zeta,h}F^{-1}$. Let g_1 be an arbitrary element of the set $H_{a_1, \dots, a_r; F}(D)$, and $G \subset H_{a_1, \dots, a_r; F}(D)$ be any open neighbourhood of g_1 . Then $F^{-1}G$ is an open set, and, in view of the inclusion $F(S) \supset H_{a_1, \dots, a_r; F}$, belongs to S . Since S is the support of $\hat{P}_{\zeta,h}$,

$$\hat{P}_{\zeta,h}F^{-1}(G) = \hat{P}_{\zeta,h}(F^{-1}G) > 0.$$

Hence, the support of $\hat{P}_{\zeta,h}F^{-1}$ contains the set $H_{a_1, \dots, a_r; F}(D)$, and thus contains its closure.

We will separate 2 cases: $r = 1$ and $r \geq 2$.

Let $r = 1$. By Lemma 26, there exists a polynomial $p(s)$ such that

$$(26) \quad \sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{4}.$$

Thus, $p(s) \neq a_1$ on K if ε is small enough. Therefore, by Lemma 26 again, there exists a polynomial $p_1(s)$ such that

$$(27) \quad \sup_{s \in K} \left| (p(s) - a_1) - e^{p_1(s)} \right| < \frac{\varepsilon}{4}.$$

For brevity, let $g_1(s) = a_1 + e^{p_1(s)}$. Obviously, $g_1(s) \in H_{a_1, F}(D)$, i.e., $g_1(s)$ is an element of the support of the measure $\hat{P}_{\zeta, h} F^{-1}$. Therefore,

$$(28) \quad \hat{P}_{\zeta, h} F^{-1}(\mathcal{G}_\varepsilon) > 0,$$

where

$$\mathcal{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - g_1(s)| < \frac{\varepsilon}{2} \right\}.$$

Hence, in virtue of Theorem 25 and 2° of Lemma 22,

$$\liminf_{N \rightarrow \infty} \frac{1}{M+1} \# \left\{ N \leq k \leq N+M : \sup_{s \in K} |F(\zeta(s+ikh)) - g_1(s)| < \frac{\varepsilon}{2} \right\} > 0.$$

This together with (26) and (27) proves the first part of the theorem.

Now, let

$$\hat{\mathcal{G}}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then, as in the proof of Theorem 1, we obtain that the set $\hat{\mathcal{G}}_\varepsilon$ is a continuity set of the measure $\hat{P}_{\zeta, h}$ for all but at most countably many $\varepsilon > 0$. Hence, by Theorem 25 and 3° of Lemma 22, the limit

$$\lim_{N \rightarrow \infty} \frac{1}{M+1} \# \left\{ N \leq k \leq N+M : \sup_{s \in K} |F(\zeta(s+ikh)) - f(s)| < \varepsilon \right\} = \hat{P}_{\zeta, h}(\hat{\mathcal{G}}_\varepsilon)$$

exists for all but at most countably many $\varepsilon > 0$. Since inequalities (26) and (27) imply the inclusion $\hat{\mathcal{G}}_\varepsilon \subset \mathcal{G}_\varepsilon$, this and (28) proves the second part of the theorem.

Now, let $r \geq 2$. Then the function $f(s)$ is an element of the support of the measure $\hat{P}_{\zeta, h} F^{-1}$. Thus, we have that $\hat{P}_{\zeta, h} F^{-1}(\hat{\mathcal{G}}_\varepsilon) > 0$, and Theorem 25 and Lemma 22 prove the both parts of the theorem. \square

5. PROOF OF THEOREM 3

Theorem 3 is an consequence of Theorem 2, the classical Rouché theorem, and conditions of the theorem.

Proof of Theorem 3. Let, for brevity,

$$\sigma_0 = \frac{\sigma_1 + \sigma_2}{2} \quad \text{and} \quad \hat{r} = \frac{\sigma_2 - \sigma_1}{2}.$$

We apply Theorem 2 with $K = \{s \in \mathbb{C} : |s - \sigma_0| \leq \hat{r}\}$ and $f(s) = s - \sigma_0$. Since $\text{Re} a_j \notin (-1/2, 1/2)$, the function $f(s) \neq a_j$ in the strip D , $j = 1, \dots, r$. Thus, the function $f(s)$ on the disc K satisfies the conditions of Theorem 2. In virtue of Theorem 2, we have that there exists a constant $c > 0$ such that, for every $\varepsilon > 0$ and sufficiently large N ,

$$(29) \quad \left\{ N \leq k \leq N+M : \sup_{s \in K} |F(\zeta(s+ikh)) - f(s)| < \varepsilon \right\} \geq cM.$$

Now, let ε satisfy the inequality

$$\varepsilon < \frac{1}{10} \sup_{|s-\sigma_0|=\hat{r}} |f(s)| = \frac{\hat{r}}{10}.$$

Then, for the above k ,

$$\sup_{|s-\sigma_0|=\hat{r}} |F(\zeta(s+ikh)) - f(s)| < \sup_{|s-\sigma_0|=\hat{r}} |f(s)|.$$

This shows that the hypotheses of the Rouché theorem, see, for example, [18], are satisfied. Therefore, the functions $F(\zeta(s+ikh))$ and $s - \sigma_0$ have the same number of zeros in the interior of the disc K . This and (29) prove the theorem. \square

Acknowledgements. The author thanks the anonymous referee for useful remarks and comments.

This research is funded by the European Social Fund according to the activity ‘Improvement of researchers’ qualification by implementing world-class R&D projects’ of Measure No. 09.3.3-LMT-K-712-01-0037.

REFERENCES

1. B. BAGCHI: *The statistical behaviour and universality properties of the Riemann zeta-function and other allied Dirichlet series*. Ph. D. Thesis, Indian Statistical Institute, Calcutta, 1981.
2. P. BILLINGSLEY: *Convergence of Probability Measures*. Willey, New York, 1968.
3. A. DUBICKAS AND A. LAURINČIKAS: *Distribution modulo 1 and the discrete universality of the Riemann zeta-function*. Abh. Math. Semin. Univ. Hamb., **86** (2016), 79–87.
4. R. GARUNKŠTIS: *The effective universality theorem for the Riemann zeta-function*. In: Proc. of the Session on Analytic Number Theory and Diophantine Equations, Bonn Math. Schriften vol. 360, Univ. Bonn, Bonn, 21 pp., 2003.
5. R. GARUNKŠTIS, A. LAURINČIKAS, K. MATSUMOTO, J. STEUDING AND R. STEUDING: *Effective uniform approximation by the Riemann zeta-function*. Publ. Math., **54** (2010), 209–219.
6. A. GOOD: *On the distribution of the values of Riemann’s zeta-function*. Acta Arith., **38** (1981), 347–388.
7. A. IVIČ: *The Theory of the Riemann Zeta-Function with Applications*. Willey, New York, 1995.
8. A. LAURINČIKAS: *Limit Theorems for the Riemann Zeta-Function*. Kluwer, Dordrecht, Boston, London, 1996.
9. A. LAURINČIKAS: *Discrete universality of the Riemann zeta-function and uniform distribution modulo 1*. St. Petersburg Math. J., **30** (2019), 103–110.

10. A. LAURINČIKAS: *Universality of the Riemann zeta-function in short intervals*. J. Number Theory, **204** (2019), 279–295.
11. A. LAURINČIKAS, K. MATSUMOTO AND J. STEUDING: *Discrete universality of L-functions of new forms. II*. Lith. Math. J., **56** (2016), 207–218.
12. R. MACAITIENĖ: *The discrete universality of the Riemann zeta-function with respect to uniformly distributed shifts*. Arch. Math. (Basel), **108** (2017), 271–281.
13. S. N. MERGELYAN: *Uniform approximations to functions of a complex variable*. Uspekhi Matem. Nauk, **7** (1952), 31–122 (in Russian).
14. H. L. MONTGOMERY: *Topics in Multiplicative Number Theory*. Lecture Notes in Math. vol. 227, Springer, Berlin, 1971.
15. S. A. MORRIS: *Pontryagin Duality and the Structure of Locally Compact Abelian Groups*. London Math. Soc. Lect. Note Ser. Vol. 29, Cambridge University Press, Cambridge, New York, Melbourne, 1971.
16. L. PAŃKOWSKI: *Joint universality for dependent L-functions*. Ramanujan J., **45** (2018), 181–195.
17. A. REICH: *Werteverteilung von Zetafunktionen*. Arch. Math., **34** (1980), 440–451.
18. D. SARASON: *Complex Function Theory*. American Math. Soc., 2007.
19. A. SHIRYAEV: *Probability*. Graduate Text in Math., 95, Springer-Verlag, New York, 1996.
20. S. M. VORONIN: *Theorem on the “universality” of the Riemann zeta-function*. Izv. Akad. Nauk SSSR, Ser. Matem., **39** (1975), 475–486 (in Russian).

Antanas Laurinčikas
Institute of Mathematics,
Faculty of Mathematics and Informatics,
Vilnius University,
Naugarduko str. 24,
LT-03225 Vilnius,
Lithuania
E-mail: antanas.laurincikas@mif.vu.lt

(Received 04.07.2019)
(Revised 10.02.2020)