VILNIUS UNIVERSITY

VAIDOTAS CHARACIEJUS

ASYMPTOTIC BEHAVIOUR OF LONG MEMORY FUNCTIONAL LINEAR PROCESSES

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Supervisor:

prof. habil. dr. Alfredas Račkauskas (Vilnius University, physical sciences, mathematics – 01 P).

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Contents

1 Introduction

Suppose that $\{X_k\} = \{X_k : k \in \mathbb{Z}\}$ is a linear process with values in a separable Hilbert space $\mathbb{H}$, i.e. $\{X_k\}$ is a sequence of $\mathbb{H}$-valued random elements such that

$$X_k = \sum_{j=0}^{\infty} a_j(\varepsilon_{k-j})$$

for each $k \in \mathbb{Z}$, where $\{a_j\} = \{a_j : j \geq 0\} \subset L(\mathbb{H})$ are bounded linear operators from $\mathbb{H}$ to $\mathbb{H}$ and $\{\varepsilon_k\} = \{\varepsilon_k : k \in \mathbb{Z}\}$ are independent and identically distributed $\mathbb{H}$-valued random elements. The interesting question is whether the asymptotic behaviour of the linear process $\{X_k\}$ differs from the asymptotic behaviour of independent and identically distributed random elements.

By asymptotic behaviour we mean the convergence in some sense of the normalised partial sums and the normalised random polygonal lines. The partial sums $\{S_n\} = \{S_n : n \geq 1\}$ are defined by

$$S_n = \sum_{k=1}^{n} X_k$$

for each $n \geq 1$ and the random polygonal lines $\{\zeta_n\} = \{\zeta_n : n \geq 1\} = \{\zeta_n(t) : t \in [0,1]\}_{n \geq 1}$ are defined by

$$\zeta_n(t) = S_{\lfloor nt \rfloor} + \{nt\} X_{\lfloor nt \rfloor + 1}$$

for each $n \geq 1$ and each $t \in [0,1]$, where $\lfloor \cdot \rfloor$ is the floor function given by $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$ for each $x \in \mathbb{R}$ and $\{x\} = x - \lfloor x \rfloor$ is the fractional part of $x \in \mathbb{R}$.

The asymptotic behaviour of the linear process $\{X_k\}$ depends on the convergence of the series

$$\sum_{j=0}^{\infty} \|a_j\|, \tag{1.1}$$

where $\|\cdot\|$ is the operator norm. If series (1.1) converges, then the asymptotic behaviour of linear processes is essentially the same as that of independent and identically distributed random elements. For example, suppose that series (1.1) converges, $\mathrm{E}\,\varepsilon_0 = 0$ and $\mathrm{E}\,\|\varepsilon_0\|^2 < \infty$, where $\|\cdot\|$ is the norm of the space $\mathbb{H}$. Then $n^{-1/2}\sum_{k=1}^{n} X_k$ converges in distribution to an $\mathbb{H}$-valued Gaussian random element with zero mean (see Merlevède, Peligrad, and Utev [42], Račkauskas and Suquet [47]). However, if series (1.1) fails to converge, the asymptotic behaviour of the linear process $\{X_k\}$ might be different than that of independent and identically distributed random elements (see Louhichi and Soulier [37], Račkauskas and Suquet [48], Characiejus and Račkauskas [6, 7]).

The main objective of this thesis is to investigate the asymptotic behaviour of the linear process $\{X_k\}$ when the series of the operator norms of $\{a_j\}$ diverges, i.e.

$$\sum_{j=0}^{\infty} \|a_j\| = \infty.$$

We establish sufficient conditions for the central limit theorem and the functional central limit theorem for a particular linear process and the Marcinkiewicz-Zygmund type weak and strong laws of large numbers for a general linear process.

Central limit theorem and functional central limit theorem

Let $(\mathbb{S}, \mathcal{S}, \mu)$ be a σ-finite measure space and $L_2(\mu) = L_2(\mathbb{S}, \mathcal{S}, \mu)$ be the real separable Hilbert space of equivalence classes of μ-almost everywhere equal square-integrable functions. In Chapter 3, we study an $L_2(\mu)$-valued linear process $\{X_k\}$ with the operators $\{a_j\}$ given by

$$a_j = (j+1)^{-D}$$

for each $j \geq 0$, where $D : L_2(\mu) \to L_2(\mu)$ is a multiplication operator such that $Df = \{d(s)f(s) : s \in \mathbb{S}\}$ for each $f \in L_2(\mu)$ with a measurable function $d : \mathbb{S} \to \mathbb{R}$ and $\{\varepsilon_j\}$ are independent and identically distributed $L_2(\mu)$-valued random elements with $\mathrm{E}\,\varepsilon_0 = 0$ and either $\mathrm{E}\,\|\varepsilon_0\|^2 < \infty$ or $\mathrm{E}\,\|\varepsilon_0\|^p < \infty$ for some $p > 2$. This linear process could serve as a model of a sequence of random functions with space varying memory and such models might be interesting in functional data analysis.

We establish sufficient conditions for the central limit theorem and the functional central limit theorem in the two cases: either $1/2 < d(s) < 1$ for each $s \in \mathbb{S}$

(Theorem 3.2 and Theorem 3.5) or $d(s) = 1$ for each $s \in \mathbb{S}$ (Theorem 3.3 and Theorem 3.6). The novelty of our results is that the normalising sequence when $1/2 < d(s) < 1$ for each $s \in \mathbb{S}$ is a sequence of operators $\{n^{-H}\} = \{n^{-H} : n \geq 1\}$, where $H : L_2(\mu) \to L_2(\mu)$ is a multiplication operator given by

$$Hf = \{[3/2 - d(s)]f(s) : s \in \mathbb{S}\}$$

for each $f \in L_2(\mu)$. The Gaussian process obtained in the functional central limit theorem when $1/2 < d(s) < 1$ for each $s \in \mathbb{S}$ generates an operator self-similar process.

Marcinkiewicz-Zygmund type laws of large numbers

In Chapter 4 we investigate an $\mathbb{H}$-valued linear process $\{X_k\}$, where $\{\varepsilon_j\}$ are independent and identically distributed $\mathbb{H}$-valued random elements with $\mathrm{E}\,\varepsilon_0 = 0$ and either $\lim_{x\to\infty} x^p \Pr\{\|\varepsilon_0\| > x\} = 0$, $\mathrm{E}\,\|\varepsilon_0\|^p < \infty$ or $\mathrm{E}[\|\varepsilon_0\|^p \log(1 + \|\varepsilon_0\|)] < \infty$ for some $1 < p < 2$. We establish sufficient conditions for the Marcinkiewicz-Zygmund type weak law of large numbers (Theorem 4.1 and Theorem 4.3) and the Marcinkiewicz-Zygmund type strong law of large numbers (Theorem 4.2 and Theorem 4.4). When the series $\sum_{j=0}^{\infty} \|a_j\|$ converges, we show that the linear process $\{X_k\}$ inherits the Marcinkiewicz-Zygmund type laws of large numbers with the same normalising sequence $\{n^{1/p}\} = \{n^{1/p} : n \geq 1\}$ from the random elements $\{\varepsilon_k\}$. However, if the series $\sum_{j=0}^{\infty} \|a_j\|$ diverges, the Marcinkiewicz-Zygmund type laws of large numbers hold with a different normalisation as we illustrate with an example in Section 4.1. We generalize the results of Louhichi and Soulier [37] in the sense that we do not assume that the distributions of $\{\varepsilon_k\}$ are α-stable.

The rest of the thesis is divided into four chapters. The purpose of Chapter 2 is to present known results about the asymptotic behaviour of linear processes and give some background on the notions of long memory and self-similarity. The central limit theorem and the functional central limit theorem is investigated in Chapter 3. In Chapter 4 we study the Marcinkiewicz-Zygmund type laws of large numbers. Finally, we give conclusions in Chapter 5.

2 Background

2.1 Linear processes

Let $\mathbb{E}$ be a separable Banach space and $L(\mathbb{E})$ be the space of bounded linear operators from $\mathbb{E}$ to $\mathbb{E}$.

Definition 2.1. A *linear process* is a sequence $\{X_k\} = \{X_k : k \in \mathbb{Z}\}$ of $\mathbb{E}$-valued random elements given by

$$X_k = \sum_{k=0}^{\infty} a_j(\varepsilon_{k-j})$$

for each $k \in \mathbb{Z}$, where $\{a_j\} = \{a_j : j \geq 0\} \subset L(\mathbb{H})$ is a sequence of bounded linear operators and $\{\varepsilon_k\} = \{\varepsilon_k : k \in \mathbb{Z}\}$ is a sequence of independent and identically distributed $\mathbb{H}$-valued random elements.

We only investigate linear processes with independent and identically distributed random elements $\{\varepsilon_k\}$. We review some results about linear processes with values in a separable Banach space $\mathbb{E}$, but the linear processes that we investigate have values in a separable Hilbert space $\mathbb{H}$.

2.2 Asymptotic behaviour of linear processes

Our target in this section is to review some known results about asymptotic behaviour of linear processes. We are interested in the convergence in distribution, almost surely and in probability of the normalised partial sums

$$b_n^{-1} S_n \tag{2.1}$$

and the convergence in distribution of the normalised random polygonal functions

$$b_n^{-1} \zeta_n, \tag{2.2}$$

where $\{b_n\} = \{b_n : n \geq 1\}$ is a normalising sequence.

2.2.1 Central limit theorem

Real linear processes

Consider a real linear process $\{X_k\}$ with $\sum_{j=0}^{\infty} a_j^2 < \infty$, $\mathrm{E}\,\varepsilon_0 = 0$ and $\mathrm{E}\,\varepsilon_0^2 < \infty$. Denote $\sigma^2 = \mathrm{E}\,\varepsilon_0^2$. Intuitively, we might expect sequence (2.1) to converge in distribution to a normally distributed random variable with the normalising sequence $\{b_n\}$ given by $b_n^2 = \mathrm{E}\,|S_n|^2$ for each $n \geq 1$. It is indeed the case, as the following theorem shows (see Theorem 18.6.5 of Ibragimov and Linnik [27] for the proof of Theorem 2.1).

Theorem 2.1. *Suppose that $\{X_k\}$ is a real linear process with $\sum_{j=0}^{\infty} a_j^2 < \infty$, $\mathrm{E}\,\varepsilon_0 = 0$ and $\mathrm{E}\,\varepsilon_0^2 < \infty$. If $\mathrm{E}\,|S_n|^2 \to \infty$ as $n \to \infty$, then*

$$\frac{S_n}{\sqrt{\mathrm{E}\,|S_n|^2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty.$$

However, it is not possible to establish the asymptotic behaviour of $\mathrm{E}\,|S_n|^2$ without any additional assumptions on the sequence $\{a_j\}$.

The asymptotic behaviour of $\mathrm{E}\,|S_n|^2$ depends on the convergence of the series

$$\sum_{j=0}^{\infty} |a_j|. \tag{2.3}$$

If series (2.3) converges and the series

$$A = \sum_{j=0}^{\infty} a_j \tag{2.4}$$

converges to a non-zero limit, then $\mathrm{E}\,|S_n|^2$ grows linearly.

Proposition 2.1. *Suppose that $\{X_k\}$ is a real linear process with $\sum_{j=0}^{\infty} |a_j| < \infty$, $A \neq 0$, $\mathrm{E}\,\varepsilon_0 = 0$ and $\mathrm{E}\,\varepsilon_0^2 < \infty$. Then*

$$\mathrm{E}\,S_n^2 \sim \sigma^2 A^2 \cdot n \quad \text{as} \quad n \to \infty.$$

Proof. Using the stationarity of $\{X_k\}$,

$$\mathrm{E}\,S_n^2 = n\Big[\,\mathrm{E}\,X_0^2 + 2\sum_{k=1}^{n-1}(1 - k/n)\,\mathrm{E}[X_0 X_k]\Big]$$

$$= n\sigma^2\Big[\sum_{j=0}^{\infty} a_j^2 + 2\sum_{k=1}^{n-1}(1-k/n)\sum_{j=0}^{\infty} a_j a_{j+k}\Big].$$

We have that

$$A^2 = \sum_{j=0}^{\infty} a_j^2 + 2\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} a_j a_{j+k}$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}(1-k/n)\sum_{j=0}^{\infty} a_j a_{j+k} = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} a_j a_{j+k}$$

since any convergent series is Cesàro summable, and the sum of the series agrees with its Cesàro sum. The proof is complete. □

If series (2.3) converges and series (2.4) converges to a non-zero limit, then sequence (2.1) converges in distribution to a normally distributed random variable with the normalising sequence $\{b_n\}$ given by $b_n = \sqrt{n}$ for each $n \geq 1$ (see also Theorem 3.11 of Phillips and Solo [46] and Theorem 4.5 of Beran, Ghosh, Feng, and Kulik [2].).

Corollary 2.1. *Suppose that* $\{X_k\}$ *is a real linear process with* $\sum_{j=0}^{\infty}|a_j| < \infty$, $A \neq 0$, $\mathrm{E}\,\varepsilon_0 = 0$ *and* $\mathrm{E}\,\varepsilon_0^2 < \infty$. *Then*

$$\frac{S_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2 A^2) \quad \textit{as} \quad n \to \infty.$$

If series (2.3) diverges, then $\mathrm{E}\,S_n^2$ might even grow faster than linearly. A simple example would be $\{a_j\}$ given by

$$a_j = (j+1)^{-\varphi} \tag{2.5}$$

for each $j \geq 0$ with $1/2 < \varphi < 1$.

Proposition 2.2. *Suppose that* $\{X_k\}$ *is a real linear process with* $\{a_j\}$ *given by* (2.5), $\mathrm{E}\,\varepsilon_0 = 0$ *and* $\mathrm{E}\,\varepsilon_0^2 < \infty$. *Then*

$$\mathrm{E}\,S_n^2 \sim \frac{\sigma^2 c(\varphi)}{(1-\varphi)(3-2\varphi)} \cdot n^{3-2\varphi} \quad \textit{as} \quad n \to \infty,$$

where

$$c(\varphi) = \int_0^{\infty} [x(x+1)]^{-\varphi}\mathrm{d}x. \tag{2.6}$$

See Giraitis, Koul, and Surgailis [19] for the proof of Proposition 2.2.

If $\{a_j\}$ is given by (2.5), then sequence (2.1) converges to a normally distributed random variable with the normalisation $\{b_n\}$ given by $b_n = n^{3/2-\varphi}$ for each $n \geq 1$.

Corollary 2.2. *Suppose that* $\{X_k\}$ *is a real linear process with* $\{a_j\}$ *given by* (2.5), $\mathrm{E}\,\varepsilon_0 = 0$ *and* $\mathrm{E}\,\varepsilon_0^2 < \infty$. *Then*

$$\frac{S_n}{n^{3/2-\varphi}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, v) \quad \text{as} \quad n \to \infty,$$

where v *is given by*

$$v = \frac{\sigma^2 c(\varphi)}{(1-\varphi)(3-2\varphi)}$$

with $c(\varphi)$ *given by* (2.6).

Linear processes with values in abstract spaces

Let us recall the definition of a Gaussian random element with values in a real separable Banach space $\mathbb{E}$. $\mathbb{E}^*$ denotes the topological dual space of $\mathbb{E}$. For details about Gaussian random elements with values in Banach spaces, see Ledoux and Talagrand [35].

Definition 2.2. An $\mathbb{E}$-valued random element X is *Gaussian* if the real random variable $f(X)$ is Gaussian for each $f \in \mathbb{E}^*$.

Consider a linear process $\{X_k\}$ with values in a separable Hilbert space $\mathbb{H}$, $\mathrm{E}\,\varepsilon_0 = 0$ and $\mathrm{E}\,\|\varepsilon_0\|^2 < \infty$, where $\|\cdot\|$ is the norm of the Hilbert space $\mathbb{H}$. It seems that there is no simple generalisation of Theorem 2.1 for linear processes with values in abstract spaces. Merlevède et al. [42] show that, without any additional assumptions on the operators $\{a_j\}$ or on the covariance operator of ε_0, the tightness of both $\{S_n/\sqrt{n} : n \geq 1\}$ and $\{S_n/\sqrt{\mathrm{E}\,\|S_n\|^2} : n \geq 1\}$ may fail and no analogue of Theorem 2.1 is possible.

However, if we assume that the series

$$\sum_{j=0}^{\infty} \|a_j\| \tag{2.7}$$

converges, essentially an analogue of Corollary 2.1 is true (see Merlevède et al. [42] for the proof of Theorem 2.2).

Theorem 2.2. *Suppose that* $\{X_k\}$ *is an* $\mathbb{H}$*-valued linear process with* $\sum_{j=0}^{\infty} \|a_j\| < \infty$, $\mathrm{E}\,\varepsilon_0 = 0$ *and* $\mathrm{E}\,\|\varepsilon_0\|^2 < \infty$. *Then*

$$\frac{S_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, AC_{\varepsilon_0}A^*) \quad \text{as} \quad n \to \infty,$$

where $\mathcal{N}$ is an $\mathbb{H}$-valued Gaussian random element, C_{ε_0} denotes the covariance operator of ε_0, $A = \sum_{j=0}^{\infty} a_j$ and A^* is the adjoint operator of A.

Račkauskas and Suquet [47] extend the result of Merlevède et al. [42] to linear processes with values in a separable Banach space $\mathbb{E}$. They establish that the linear process inherits its asymptotic behaviour from $\{\varepsilon_k\}$ if series (2.7) converges.

Theorem 2.3. *Suppose that $\{X_k\}$ is an $\mathbb{E}$-valued linear process with $\sum_{j=0}^{\infty} \|a_j\| < \infty$. If*

$$\frac{\varepsilon_1 + \ldots + \varepsilon_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, C_{\varepsilon_0}) \quad as \quad n \to \infty,$$

then

$$\frac{S_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, AC_{\varepsilon_0}A^*) \quad as \quad n \to \infty,$$

where $\mathcal{N}$ is an $\mathbb{E}$-valued Gaussian random element, C_{ε_0} denotes the covariance operator of ε_0, $A = \sum_{j=0}^{\infty} a_j$ and A^* is the adjoint operator of A.

Remark 2.1. If $\{\varepsilon_k\}$ are independent and identically distributed random elements with values in the separable Hilbert space $\mathbb{H}$, then $\mathrm{E}\,\varepsilon_0 = 0$ and $\mathrm{E}\,\|\varepsilon_0\|^2 < \infty$ implies that $\{\varepsilon_k\}$ satisfies the central limit theorem in $\mathbb{H}$. However, in some separable Banach spaces no integrability condition ensures that $\{\varepsilon_k\}$ satisfies the central limit theorem. For this reason, the assumptions of Theorem 2.2 and Theorem 2.3 are different (for more details, see Ledoux and Talagrand [35]).

In Chapter 3 we investigate the central limit theorem for a functional linear process with values in a particular Hilbert space when series (2.7) diverges. The case we investigate shows that the normalizing sequence might be a sequence of operators when series (2.7) diverges.

2.2.2 Functional central limit theorem

Real linear process

Consider a real linear process $\{X_k\}$ with $\sum_{j=0}^{\infty} a_j^2 < \infty$, $\mathrm{E}\,\varepsilon_0 = 0$ and $0 < \mathrm{E}\,\varepsilon_0^2 < \infty$. As in the case of the central limit theorem, the asymptotic behaviour of sequence (2.2) depends on the convergence of the series (2.3). Depending on the asymptotic behaviour of the variance of the partial sums, the limit can be the Wiener process or the fractional Brownian motion.

Let us recall the definitions of the Wiener process and the fractional Brownian motion.

Definition 2.3. The *Wiener process* $\{W(t) : t \geq 0\}$ is a real-valued Gaussian random process such that

(i) $\mathrm{E}\, W(t) = 0$ for each $t \geq 0$;

(ii) $\mathrm{E}[W(s)W(t)] = \min\{s, t\}$ for $s \geq 0$ and $t \geq 0$;

(iii) $\Pr\{W \in C[0, \infty)\} = 1$.

Definition 2.4. The *fractional Brownian motion* $\{B_H(t) : t \geq 0\}$ with the self-similarity parameter $H \in (0, 1)$ (or the Hurst parameter) is a real-valued Gaussian process such that

(i) $\mathrm{E}\, B_H(t) = 0$ for each $t \geq 0$;

(ii) $\mathrm{E}[B_H(s)B_H(t)] = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s - t|^{2H})$ for $s \geq 0$ and $t \geq 0$;

(iii) $\Pr\{B_H \in C[0, \infty)\} = 1$.

Although the fractional Brownian motion was introduced by Kolmogorov [30], it was Mandelbrot and Van Ness [39] who recognized the relevance of this random process and gave this process the name by which it is known today. Let us observe that the fractional Brownian motion with $H = 1/2$ is the Wiener process. See Chapter 6 of Resnick [50] for more details about the Wiener process and see Chapter 3 of Beran et al. [2] or Chapter 7 of Samorodnitsky and Taqqu [53] for more details about the fractional Brownian.

We begin with the case when series (2.3) converges and the series A given by (2.4) converges to a non-zero limit (see Wang, Lin, and Gulati [57] or Merlevède, Peligrad, and Utev [43] for the proof of Theorem 2.4).

Theorem 2.4. *Suppose that* $\{X_k\}$ *is a real linear process with* $\sum_{j=0}^{\infty} |a_j| < \infty$, $A \neq 0$, $\mathrm{E}\, \varepsilon_0 = 0$ *and* $\mathrm{E}\, \varepsilon_0^2 < \infty$. *Then*

$$b_n^{-1} \zeta_n \xrightarrow{\mathcal{D}} W$$

in $C[0,1]$ *as* $n \to \infty$ *with* $\{b_n\}$ *given by*

$$b_n = A \cdot \sqrt{n}.$$

As we have already seen in Proposition 2.2, if series (2.3) diverges, the variance of the partial sums can grow faster than linearly. This changes not only the normalizing sequence $\{b_n\}$, but also the limit of sequence (2.2). The following result was first proven by Davydov [13] under the assumption of $\mathrm{E}\,\varepsilon_0^{2k} < \infty$ for $k \geq 2$. It is possible to prove the following result under the assumption of $\mathrm{E}\,\varepsilon_0^2 < \infty$ (see Theorem 4.6 of Beran et al. [2] or Theorem 1 of Konstantopoulos and Sakhanenko [31] for the proof of Theorem 2.5).

Theorem 2.5. *Suppose that* $\{X_k\}$ *is a real linear process with* $\{a_j\}$ *given by* (2.5), $\mathrm{E}\,\varepsilon_0 = 0$ *and* $\mathrm{E}\,\varepsilon_0^2 < \infty$. *Then*

$$b_n^{-1}\zeta_n \xrightarrow{\mathcal{D}} B_{3/2-\varphi}$$

in $C[0,1]$ *as* $n \to \infty$ *with*

$$b_n^2 = \frac{\sigma^2 c(\varphi)}{(1-\varphi)(3-2\varphi)} \cdot n^{3-2\varphi},$$

where $c(\varphi)$ *is given by* (2.6) *and* $B_{3/2-\varphi}$ *is the fractional Brownian motion with the self-similarity paramter equal to* $3/2 - \varphi$.

Linear processes with values in abstract spaces

Before we review the functional central limit theorems for linear processes with values in abstract spaces, we need to introduce more general definitions of a Gaussian random process and the Wiener process.

Definition 2.5. An $\mathbb{E}$-valued random process $\{\xi(t) : t \in T\}$, indexed by some set T, is Gaussian if

$$\sum_{i=1}^{n} \alpha_i \xi(t_i)$$

is an $\mathbb{E}$-valued Gaussian random element for any $n \geq 1$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $t_1, \ldots, t_n \in T$.

For details about Gaussian random processes with values in Banach spaces, see Ledoux and Talagrand [35].

Definition 2.6. An $\mathbb{E}$-*valued Wiener process* $\{W_Q(t) : t \geq 0\}$ is an $\mathbb{E}$-valued centred Gaussian random process with independent increments such that $W_Q(t) - W_Q(s)$ has the same distribution as $|t-s|^{1/2}G$ for $t > s \geq 0$, where G is an $\mathbb{E}$-valued centred Gaussian random element with the covariance operator Q.

See Van Thu [55] for alternative definitions of an $\mathbb{E}$-valued Wiener process.

Račkauskas and Suquet [47] prove that the $\mathbb{E}$-valued linear process inherits its asymptotic behaviour from $\{\varepsilon_k\}$ if series (2.7) converges.

Theorem 2.6. *Suppose that $\{X_k\}$ is an $\mathbb{E}$-valued linear process with $\sum_{j=0}^{\infty} \|a_j\| < \infty$ and*

$$\frac{\varepsilon_1 + \ldots + \varepsilon_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, C_{\varepsilon_0}) \quad as \quad n \to \infty.$$

Then

$$\frac{\zeta_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} W_{AC_{\varepsilon_0}A^*}$$

in $C([0,1];\mathbb{E})$ as $n \to \infty$, where $W_{AC_{\varepsilon_0}A^*}$ is an $\mathbb{E}$-valued Wiener process, C_{ε_0} denotes the covariance operator of ε_0, $A = \sum_{j=0}^{\infty} a_j$ and A^* is the adjoint operator of A.

Duncan, Pasik-Duncan, and Maslowski [15] generalize the definition the fractional Brownian motion. They establish the existence of the fractional Brownian motion with values in a separable Hilbert space $\mathbb{H}$. Suppose that $Q \in L(\mathbb{H})$ is a non-negative, self-adjoint and trace class operator. Let $\operatorname{Cov}[X, Y]$ denote the covariance operator of two $\mathbb{H}$-valued random elements X and Y. Duncan et al. [15] give the following definition of the fractional Q-Brownian motion.

Definition 2.7. The $\mathbb{H}$-*valued fractional Q-Brownian motion* $\{B_{H,Q}(t) : t \geq 0\}$ with the Hurst parameter $1/2 < H < 1$ is an $\mathbb{H}$-valued Gaussian random process such that

(i) $\mathrm{E}\, B_{H,Q}(t) = 0$ for each $t \geq 0$;

(ii) $\operatorname{Cov}[B_{H,Q}(s), B_{H,Q}(t)] = 2^{-1}(|s|^{2H} + |t|^{2H} - |s-t|^{2H})Q$ for all $0 \leq s < t$;

(iii) $\Pr\{B_{H,Q} \in C([0,\infty);\mathbb{H})\} = 1$.

Račkauskas and Suquet [47] define the $\mathbb{H}$-valued fractional Brownian motion with the Hurst parameter $H \in L(\mathbb{H})$. Let $H \in L(\mathbb{H})$ and $Q \in L(\mathbb{H})$ be non-negative operators and suppose that Q is trace class.

Definition 2.8. The $\mathbb{H}$-*valued operator fractional* Q-*Brownian motion* with the Hurst parameter H is an $\mathbb{H}$-valued Gaussian random process such that

(i) $\operatorname{E} B_{H,Q}(t) = 0$ for each $t \geq 0$;

(ii) $\operatorname{Cov}[B_{H,Q}(s), B_{H,Q}(t)] = 2^{-1}(|s|^{2H} + |t|^{2H} - |s-t|^{2H})Q$ for all $s \geq 0$ and $t \geq 0$.

Račkauskas and Suquet [48] prove the existence of an $\mathbb{H}$-valued operator fractional Q-Brownian motion only under additional assumptions on the operators H and Q (they assume that H is a self-adjoint operator such that $\frac{1}{2}I < H < I$ and the operator H commutes with the operator Q). They also establish that the $\mathbb{H}$-valued operator fractional Q-Brownian motion has a continuous version.

Consider an $\mathbb{H}$-valued linear process $\{X_k\}$ with $\{a_j\}$ given by

$$a_0 = I, \qquad a_j = j^{-T} \qquad \text{for} \qquad j \geq 1, \tag{2.8}$$

where $T \in L(\mathbb{H})$ satisfies $1/2I < T < I$, $\operatorname{E}\varepsilon_0 = 0$ and $\operatorname{E}\|\varepsilon_0\|^2 < \infty$. Let Q be a covariance operator of ε_0. Assume that T commutes with Q. Račkauskas and Suquet [48] show that the series

$$\sum_{j=0}^{\infty} \|a_j\| = \infty$$

and sequence (2.2) converges in distribution to an $\mathbb{H}$-valued operator fractional Q-Brownian motion with the Hurst parameter $H = 3/2I - T$.

Theorem 2.7. *Suppose that* $\{X_k\}$ *is an* $\mathbb{H}$*-valued linear process with* $\{a_j\}$ *given by* (2.8)*,* $\operatorname{E}\varepsilon_0 = 0$*,* $\operatorname{E}\|\varepsilon_0\|^2 < \infty$ *and the covariance operator* Q *commutes with* T*. Then*

$$c(T)n^{-H}\zeta_n \xrightarrow{\mathcal{D}} B_{H,Q}$$

in $C([0,1];\mathbb{H})$ *as* $n \to \infty$*, where* $B_{H,Q}$ *is an operator fractional* Q*-Brownian motion with the operator Hurst parameter* $H = 3/2I - T$ *and bounded linear operator* $c(T)$*.*

See Račkauskas and Suquet [48] for the expression of the operator $c(T)$.

In Chapter 3 we investigate the case when the series of operator norms of $\{a_j\}$ diverges and T not necessarily commutes with Q. When T does not commute with Q, the random polygonal lines converge in distribution to a different Gaussian process than the operator fractional Q-Brownian motion.

2.2.3 Law of large numbers

The law of large numbers is a well-know result in the probability theory, but we state both the strong and weak law of large numbers here in order to emphasize different assumptions of the weak and the strong law of large numbers.

Theorem 2.8 (Weak law of large numbers)**.** *Suppose that* $\{\xi_n : n \geq 1\}$ *are independent and identically distributed random variables and set* $S_n = \sum_{k=1}^{n} \xi_k$. *There exists a real sequence* $\{\mu_n : n \geq 1\}$ *such that*

$$\frac{S_n}{n} - \mu_n \to 0 \tag{2.9}$$

in probability as $n \to \infty$ *if and only if* $x \Pr\{|\xi_0| > x\} \to 0$ *as* $x \to \infty$. *In this case* (2.9) *holds with* $\mu_n = \mathrm{E}[\xi_0 I_{\{|\xi_0| \leq n\}}]$.

See Feller [17] and Resnick [51] for the proof of Theorem 2.8.

Theorem 2.9 (Strong law of large numbers)**.** *Suppose that* $\{\xi_n : n \geq 1\}$ *are independent and identically distributed random variables and set* $S_n = \sum_{k=1}^{n} \xi_k$. *There exists* $c \in \mathbb{R}$ *such that*

$$\frac{S_n}{n} \to c$$

almost surely as $n \to \infty$ *if and only if* $\mathrm{E}\,|\xi_0| < \infty$ *in which case* $c = \mathrm{E}\,\xi_0$.

See Resnick [51] for the proof of Theorem 2.9.

Phillips and Solo [46] established the strong law of large numbers for real linear processes. They proved the following two theorems.

Theorem 2.10. *Suppose that*

$$\sum_{j=1}^{\infty} |ja_j|^2 < \infty,$$

$\{\varepsilon_k\}$ *are independent and identically distributed with* $\mathrm{E}\,\varepsilon_0^2 < \infty$ *and zero means. Then*

$$\frac{S_n}{n} \to 0$$

almost surely as $n \to \infty$.

Theorem 2.11. *Suppose that*

$$\sum_{j=1}^{\infty} j|a_j| < \infty,$$

$\{\varepsilon_k\}$ are independent and identically distributed with $\mathrm{E}\,|\varepsilon_0| < \infty$ and zero mean. Then

$$\frac{S_n}{n} \to 0$$

almost surely as $n \to \infty$.

There are generalizations of both the weak law of large numbers and the strong law of large numbers.

Theorem 2.12 (Marcinkiewicz-Zygmund type weak law of large numbers)**.** *Suppose that $\{\xi_n : n \geq 1\}$ are independent and identically distributed random variables and set $S_n = \sum_{k=1}^{n} \xi_k$. Fix any $p \in (0, 2)$ and $c \in \mathbb{R}$. Then*

$$\frac{S_n}{n^{1/p}} \to c$$

in probability as $n \to \infty$ if and only if the following conditions hold as $r \to \infty$, depending on the value of p:

$p < 1$: $r^p \Pr\{|\xi_1| > r\} \to 0$ *and* $c = 0$;

$p = 1$: $r \Pr\{|\xi_1| > r\} \to 0$ *and* $\mathrm{E}[\xi_1 I_{\{|\xi_1| \leq r\}}] \to c$;

$p > 1$: $r^p \Pr\{|\xi_1| > r\} \to 0$ *and* $\mathrm{E}\,\xi_1 = c = 0$.

Theorem 2.13 (Marcinkiewicz-Zygmund strong law of large numbers)**.** *Suppose that $\{\xi_n : n \geq 1\}$ are independent and identically distributed random variables and set $S_n = \sum_{k=1}^{n} \xi_k$. Fix any $p \in (0, 2)$. Then*

$$\frac{S_n}{n^{1/p}}$$

converges almost surely as $n \to \infty$ if and only if $\mathrm{E}\,|\xi_1|^p < \infty$ and either $p \leq 1$ or $\mathrm{E}\,\xi_1 = 0$. In that case the limit equals $\mathrm{E}\,\xi_1$ for $p = 1$ and is otherwise 0.

See Kallenberg [29] for the proofs of Theorem 2.12 and Theorem 2.13.

Louhichi and Soulier [37] investigate the Marcinkiewicz-Zygmund type strong law of large numbers for real linear processes when the series $\sum_{j=0}^{\infty} |a_j|$ not necessarily converges. Suppose that $\{\varepsilon_k\}$ are independent and identically distributed symmetric α-stable variables with $1 < \alpha < 2$ or uncorrelated with finite variance. The latter case is refered to as the case $\alpha = 2$ for convenience. Let $\{a_j\}$ be a sequence of real numbers such that $\sum_{j=0}^{\infty} |a_j|^{\alpha} < \infty$. Louhichi and Soulier [37] prove the following theorem.

Theorem 2.14. *Assume that there exists a real $s \in [1, \alpha)$ such that $\sum_{j=0}^{\infty} |a_j|^s < \infty$. Then for all p such that $1/p > 1 - 1/s + 1/\alpha$,*

$$\frac{S_n}{n^{1/p}} \to 0$$

almost surely as $n \to \infty$.

In Chapter 4 we investigate the Marcinkiewicz-Zygmund type weak and strong laws of large numbers for general linear processes with values the space $\mathbb{H}$. We make assumptions of $\mathrm{E}\,\varepsilon_0 = 0$ and either $\lim_{x\to\infty} x^p \Pr\{\|\varepsilon_0\| > x\} = 0$, $\mathrm{E}\,\|\varepsilon_0\|^p < \infty$ or $\mathrm{E}[\|\varepsilon_0\|^p \log(1 + \|\varepsilon_0\|)] < \infty$ for some $1 < p < 2$. So we generalize Theorem 2.14 in the sense that we do not assume that the distributions of $\{\varepsilon_k\}$ are α-stable.

The following two theorems are generalizations of the Marcinkiewicz-Zygmund type laws of large numbers to separable Hilbert spaces.

Theorem 2.15. *Let $1 \le p < 2$. Suppose that $\{\xi_n : n \ge 1\}$ are independent and identically distributed symmetric random elements with values in a separable Hilbert space $\mathbb{H}$. Then*

$$\frac{\sum_{k=1}^{n} \xi_k}{n^{1/p}} \to 0$$

in probability as $n \to \infty$ if and only if $\lim_{x\to\infty} x^p \Pr\{\|\xi_0\| > x\} = 0$.

Theorem 2.16. *Let $0 < p < 2$. Suppose that $\{\xi_n : n \ge 1\}$ are independent and identically distributed random elements with values in a separable Hilbert space $\mathbb{H}$. Then*

$$\frac{\sum_{k=1}^{n} \xi_k}{n^{1/p}} \to 0$$

almost surely as $n \to \infty$ if and only if $\mathrm{E}\,\|\xi_1\|^p < \infty$ and $\mathrm{E}\,\xi_1 = 0$ if $p \ge 1$.

See Ledoux and Talagrand [35] for the proofs of Theorem 2.15 and Theorem 2.16.

2.3 Memory

There is evidence that long memory processes occur quite frequently in various fields, such as finance, econometrics, internet modeling, hydrology, climates studies, linguistics, DNA sequencing (we refer to Samorodnitsky [52] for a review of the notion of long memory; for probabilistic foundations, statistical methods, and applications, see Giraitis et al. [19], Beran [1] and Palma [45]).

There are many definitions of the memory of a random process and, unfortunately, the definitions are not equivalent (Guégan [21] mentions 11 different definitions). We restrict our attention to stationary and strictly stationary sequences of random variables.

Definition 2.9. A sequence $\{X_k\} = \{X_k : k \in \mathbb{Z}\}$ of random variables is *stationary* if

(i) $\mathrm{E}\, X_k^2 < \infty$ for each $k \in \mathbb{Z}$;

(ii) $\mathrm{E}\, X_k = m$ for each $k \in \mathbb{Z}$, where $m \in \mathbb{R}$;

(iii) $\mathrm{Cov}[X_r, X_s] = \mathrm{Cov}[X_{r+t}, X_{s+t}]$ for all $r, s, t \in \mathbb{Z}$.

Remark 2.2. If $\{X_k\}$ is stationary, then $\mathrm{Cov}[X_r, X_s] = \mathrm{Cov}[X_0 X_{s-r}]$ for all $s, t \in \mathbb{Z}$ and it makes sense to investigate $\mathrm{Cov}[X_0 X_h]$ for $h \geq 0$ instead of $\mathrm{Cov}[X_r, X_s]$ for $r, s \in \mathbb{Z}$.

Definition 2.10. A sequence of random variables $\{X_k : k \in \mathbb{Z}\}$ is said to be *strictly stationary* if the joint distributions of $(X_{t_1}, \ldots, X_{t_k})$ and $(X_{t_1+h}, \ldots, X_{t_k} + h)$ are the same for all $k \geq 1$ and for all $t_1, \ldots, t_k, h \in \mathbb{Z}$.

If we are considering stationary sequences of random variables, we can define the memory of a random process in terms of the asymptotic behaviour of the sequence of covariances.

Definition 2.11. A stationary sequence of random variables $\{X_k\}$ has *long memory* if the series

$$\sum_{j=0}^{\infty} |\,\mathrm{Cov}[X_0 X_j]| \tag{2.10}$$

diverges. If series (2.10) converges, then the sequence $\{X_k\}$ has *short memory*.

Clearly, a sequence of independent and identically distributed random variables has short memory. An example of a sequence of random variables that has long memory is not straightforward. Two well-known examples are the fractional Gaussian noise and the fractional ARIMA process.

Definition 2.12. The *fractional Gaussian noise* is a sequence $\{\xi_k\} = \{\xi_k : k \geq 0\}$ of the increments of the fractional Brownian motion $\{B_H(t) : t \geq 0\}$ given by

$$\xi_k = B_H(k) - B_H(k-1)$$

for $k \geq 0$, where $H \in (0,1)$ is the self-similarity parameter.

The fractional Gaussian noise has the following properties:

(i) $\mathrm{Corr}[\xi_{j+h}, \xi_j] \sim H(2H-1)h^{-2(1-H)}$ as $h \to \infty$ for $H \in (0,1)$ and $H \neq 1/2$;

(ii) $\mathrm{Var}\Big[\sum_{k=1}^{n} \xi_k\Big] = \mathrm{E}[B_H^2(1)] \cdot n^{2H}$ for $H \in (0,1)$.

If $1/2 < H < 1$, the fractional Gaussian noise has long memory.

Definition 2.13. Let $-1/2 < \psi < 1/2$. The *fractional ARIMA*$(0,\psi,0)$ *process* with the parameter ψ is a real linear process with $\{a_j\}$ given by

$$a_j = \frac{\Gamma(j+\psi)}{\Gamma(j+1)\Gamma(\psi)} = \prod_{0<k\leq j} \frac{k-1+\psi}{k}$$

for each $j \geq 0$, where Γ is the gamma function.

The fractional ARIMA$(0,\psi,0)$ process was introduced independently by Granger and Joyeux [20] and Hosking [25]. We have that (see Chapter 13 of Brockwell and Davis [5])

$$\mathrm{Corr}[X_0, X_h] \sim \frac{\Gamma(1-\psi)}{\Gamma(\psi)} \cdot h^{2\psi-1} \quad \text{as} \quad h \to \infty,$$

where $\{X_k\}$ is the fractional ARIMA$(0,\psi,0)$ process with $-1/2 < \psi < 1/2$. Thus the fractional ARIMA$(0,\psi,0)$ with $0 < \psi < 1/2$ has long memory.

The idea behind Definition 2.11 is that the absolute summability of the autocovariances $\{\mathrm{Cov}[X_0, X_j] : j \geq 0\}$ implies at most linear growth of the variance of the partial sums.

Proposition 2.3. *Suppose that* $\{X_k\}$ *is a stationary sequence that has short memory. Then* $\mathrm{Var}\, S_n = O(n)$ *as* $n \to \infty$*. Furthermore, if*

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} (1-k/n)\,\mathrm{Cov}[X_0, X_k] \neq -\frac{1}{2}\mathrm{Var}\, X_0,$$

then $\operatorname{Var} S_n \sim c \cdot n$ *as* $n \to \infty$, *where* c *is a positive constant.*

Proof. . Using the stationarity of the sequence $\{X_k\}$,

$$\frac{1}{n}\operatorname{Var} S_n = \operatorname{Var} X_0 + 2\sum_{k=1}^{n-1}(1-k/n)\operatorname{Cov}[X_0, X_k].$$

Similarly, as in the proof of Proposition 2.1, we have that

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}(1-k/n)\operatorname{Cov}[X_0, X_k] = \sum_{k=1}^{\infty}\operatorname{Cov}[X_0, X_k]$$

since any convergent series is Cesàro summable, and the sum of the series agrees with its Cesàro sum. □

The following definition of the memory of a random process was proposed by Cox [10] and it is relevant to this thesis since we are investigating the functional central limit theorem.

Definition 2.14. Let $\{a_n : n \geq 1\}$ and $\{b_n : n \geq 1\}$ be real sequences such that $b_n \to \infty$ as $n \to \infty$. A strictly stationary sequence of random variables $\{\xi_k : k \in \mathbb{Z}\}$ has *long memory* if the finite-dimensional distributions of the random processes

$$\left\{ b_n^{-1}\sum_{k=1}^{\lfloor nt \rfloor}(\xi_k - a_n) : t \geq 0 \right\}$$

converge weakly to the finite-dimensional distributions of the random process with dependent increments. If the increments of the limit process are independent, the sequence $\{\xi_k : k \in \mathbb{Z}\}$ has *short memory.*

The long memory (short memory) defined in Definition 2.11 is sometimes called *covariance long memory* (*covariance short memory*) and the long memory defined in Definition 2.14 is sometimes called *distributional long memory* (*distributional short memory*) to make a distinction between these two definitions.

Definition 2.11 and Definition 2.14 are not equivalent. Consider, for example, a real linear process with $\{a_j\}$ given by $a_j = (j+1)^{-1}$ for each $j \geq 0$, $\operatorname{E}\varepsilon_0 = 0$ and $\operatorname{E}\varepsilon_0^2 < \infty$. Then it has covariance long memory since

$$\sum_{h=0}^{\infty}|\operatorname{Cov}[X_0, X_h]| = \infty,$$

but distributional short memory since the finite-dimensional distributions of the processes

$$\left\{(\sigma\cdot\sqrt{n}\log n)^{-1}\sum_{k=1}^{\lfloor nt\rfloor}X_k : t\geq 0\right\}$$

converge weakly to the finite dimensional distributions of the Wiener process, i.e. a process with independent increments.

Louhichi and Soulier [37] proposed a definition of long memory based on the rate of convergence in the Marcinkiewicz-Zygmund type strong law of large numbers.

Definition 2.15. Let $\{\xi_k : k\in\mathbb{Z}\}$ be a strictly stationary sequence of random variables with zero means and $\mathrm{E}\,|\xi_0|^q<\infty$ for some $q\in[1,2]$. The sequence $\{\xi_k : k\in\mathbb{Z}\}$ has *short memory* if

$$\frac{\sum_{k=1}^{n}\xi_k}{n^{1/p}}\to 0$$

almost surely as $n\to\infty$ for all $0<p\leq q$ and $p<2$. The sequence $\{\xi_k\}$ has *long memory* otherwise.

2.4 Self-similarity

Self-similar processes are random processes that are invariant in distribution under suitable scaling of time and space. More precisely, let $\xi=\{\xi(t) : t\geq 0\}$ be an $\mathbb{R}^q$-valued stochastic process defined on some probability space $(\Omega,\mathcal{F},P)$. The process ξ is said to be self-similar if for any $a>0$ there exists $b>0$ such that

$$\{\xi(at) : t\geq 0\}\overset{fdd}{=}\{b\xi(t) : t\geq 0\},$$

where $\overset{fdd}{=}$ denotes the equality of the finite-dimensional distributions.

Self-similar processes were first studied rigorously by Lamperti [33]. Well-known examples are the Wiener process and the fractional Brownian motion with the self-similarity parameter $0<H<1$ (in these cases b is equal to $a^{1/2}$ and a^H respectively). We refer to Embrechts and Maejima [16] for the current state of knowledge about self-similar processes and their applications.

Laha and Rohatgi [32] introduced *operator* self-similar processes taking values in $\mathbb{R}^q$. They extended the notion of self-similarity to allow scaling by a class of matrices. Such processes were later studied by Hudson and Mason [26], Maejima and Mason [38], Lavancier, Philippe, and Surgailis [34] and Didier and Pipiras [14] among others.

Matache and Matache [41] consider and study operator self-similar processes valued in (possibly infinite-dimensional) Banach spaces. Recall that $\mathbb{E}$ denotes a Banach space and $L(\mathbb{E})$ is the algebra of all bounded linear operators from $\mathbb{E}$ to $\mathbb{E}$. Matache and Matache [41] give the following definition.

Definition 2.16. An operator self-similar process is a random process $\xi = \{\xi(t) : t \geq 0\}$ on $\mathbb{E}$ such that there is a family $\{T(a) : a > 0\}$ in $L(\mathbb{E})$ with the property that for each $a > 0$,

$$\{\xi(at) : t \geq 0\} \overset{fdd}{=} \{T(a)\xi(t) : t \geq 0\}.$$

The family $\{T(a) : a > 0\}$ is called the scaling family of operators. If operators $\{T(a) : a > 0\}$ have the particular form $T(a) = a^G I$, where G is some fixed scalar and I is an identity operator, then a random process is called self-similar instead of operator self-similar.

3 Central limit theorem and functional central limit theorem

In this chapter we investigate a linear process $\{X_k\} = \{X_k : k \in \mathbb{Z}\}$ with values in a separable Hilbert space $L_2(\mu)$ defined by

$$X_k = \sum_{j=0}^{\infty} a_j(\varepsilon_{k-j}) \tag{3.1}$$

for each $k \in \mathbb{Z}$ with $\{a_j\}$ given by

$$a_j = (j+1)^{-D} \tag{3.2}$$

for $j \geq 0$, where $D : L_2(\mu) \to L_2(\mu)$ is a multiplication operator such that $Df = \{d(s)f(s) : s \in \mathbb{S}\}$ for each $f \in L_2(\mu)$ and $d : \mathbb{S} \to \mathbb{R}$ is a measurable function. We assume that $\{\varepsilon_k\} = \{\varepsilon_k : k \in \mathbb{Z}\}$ are independent and identically distributed $L_2(\mu)$-valued random elements with $\mathrm{E}\,\varepsilon_0 = 0$ and either $\mathrm{E}\,\|\varepsilon_0\|^2 < \infty$ or $\mathrm{E}\,\|\varepsilon_0\|^p < \infty$ for some $p > 2$. We establish sufficient conditions for the central limit theorem and the functional central limit theorem for $\{X_k\}$.

3.1 Preliminaries

3.1.1 Construction of linear processes

There are two approaches to construct $\{X_k\}$ with values in $L_2(\mu)$. The first approach is to define $\{X_k\}$ as random processes with space varying memory and square μ-integrable sample paths. The second approach is to define $L_2(\mu)$-valued linear process with $\{a_j\}$ given by (3.2) and to investigate the convergence of series (3.1). We present both of these two approaches.

First approach

Let $\{\varepsilon_k\} = \{\varepsilon_k(s) : s \in \mathbb{S}\}_{k\in\mathbb{Z}}$ be independent and identically distributed measurable random processes defined on the probability space $(\Omega, \mathcal{F}, P)$, i.e. $\{\varepsilon_k\}$ are $\mathcal{F} \otimes \mathcal{S}$-measurable functions $\varepsilon_k : \Omega \times \mathbb{S} \to \mathbb{R}$. We require that $\mathrm{E}\,\varepsilon_0(s) = 0$ and $\mathrm{E}\,\varepsilon_0^2(s) < \infty$ for each $s \in \mathbb{S}$ and denote

$$\sigma(r, s) = \mathrm{E}[\varepsilon_0(r)\varepsilon_0(s)], \quad \sigma^2(s) = \mathrm{E}\,\varepsilon_0^2(s), \quad r,\, s \in \mathbb{S}.$$

Define stochastic processes $\{X_k\} = \{X_k(s) : s \in \mathbb{S}\}_{k\in\mathbb{Z}}$ by setting

$$X_k(s) = \sum_{j=0}^{\infty} (j+1)^{-d(s)} \varepsilon_{k-j}(s) \tag{3.3}$$

for each $s \in \mathbb{S}$ and each $k \in \mathbb{Z}$. It follows from Kolmogorov's three-series theorem that $d(s) > 1/2$ is a necessary and sufficient condition for the almost sure convergence of series (3.3) (see Chapter 2 of Shiryaev [54] for Kolmogorov's three-series theorem).

If $\mathrm{E}\,\varepsilon_0(s) \neq 0$, then the sequence $\{X_k(s)\}$ for $s \in \mathbb{S}$ can only have short memory, since then the series (3.3) converges almost surely if and only if $d(s) > 1$ and absolute summability of $\{a_j\}$ implies absolute summability of the autocovariances of a linear process (see, for example, Hamilton [23], p. 70).

The sequence $\{X_k(s)\}$ for each $s \in \mathbb{S}$ is essentially similar to the fractional ARIMA$(0, 1 - d(s), 0)$ process (see Definition 2.13). $\{X_k(s)\}$ is a real linear processes

$$X_k(s) = \sum_{j=0}^{\infty} a_j(s) \varepsilon_{k-j}(s)$$

with the sequence $\{a_j(s)\}$ given by $a_j(s) = (j + 1)^{-d(s)}$. The application of Stirling's formula to the coefficients of the fractional ARIMA$(0, 1 - d(s), 0)$ yields the following relation:

$$\frac{\Gamma(j + 1 - d(s))}{\Gamma(j + 1)\Gamma(1 - d(s))} \sim \frac{j^{-d(s)}}{\Gamma(1 - d(s))}$$

as $j \to \infty$.

It is possible to consider more general case of a linear process $\{X_k(s)\}$. For example, $a_j(s) \sim j^{-d(s)}$ as $j \to \infty$. But our aim is to investigate space varying memory and we want to avoid any unnecessary technical difficulties.

The growth rate of the partial sums $\{\sum_{k=1}^{n} X_k(s)\}$ depends on $d(s)$. Viewing $\mathbb{S}$ as the set of space indexes and $\mathbb{Z}$ as the set of time indexes, we thus have a functional process $\{X_k\}$ with space varying memory. Such sequences of random processes could serve as a model in functional data analysis (we refer to Ramsay and Silverman [49] and Horváth and Kokoszka [24] for an introduction to functional data analysis, for the theory of linear processes in function spaces, see Bosq [4] and Mas and Pumo [40]).

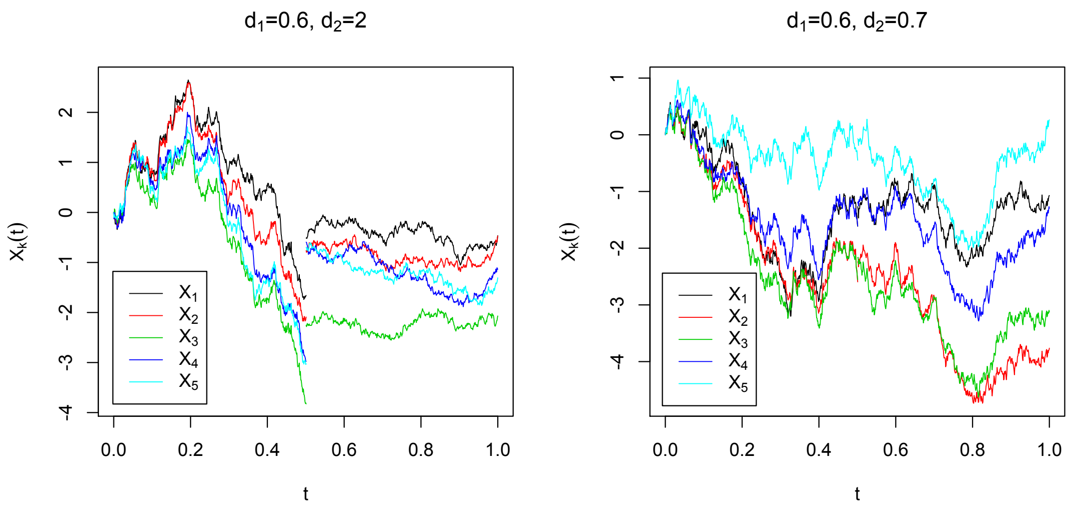

Figure 3.1: Simulated sample paths of the random processes $\{X_k\}$

Figure 3.1 shows simulated sample paths of the random processes of the sequence $\{X_k\}$. The sequence $\{\varepsilon_k\} = \{\varepsilon_k(t) : t \in [0,1]\}$ was assumed to be a sequence of independent and identically distributed standard Wiener processes on the interval $[0,1]$ and the function $d : [0,1] \to (1/2, +\infty)$ was assumed to be a step function $d(t) = d_1\chi_{[0,1/2)}(t) + d_2\chi_{[1/2,1]}(t)$, where χ_A is the indicator function of A. The simulated sample paths for 5 consecutive elements of the sequence $\{X_k\}$ were plotted. The procedure was completed for two different sets of the values of d_1 and d_2 ($d_1 = 0.6, d_2 = 2$ and $d_1 = 0.6, d_2 = 0.7$).

We denote

$$\gamma_h(r,s) = \mathrm{E}[X_0(r)X_h(s)] \qquad \text{and} \qquad \gamma_h(s) = \mathrm{E}[X_0(s)X_h(s)]$$

for $r, s \in \mathbb{S}$ and $h \geq 1$. For fixed $r, s \in \mathbb{S}$, the sequences $\{X_k(r)\}$ and $\{X_k(s)\}$ are stationary sequences of random variables with zero means and cross-covariance

$$\gamma_h(r,s) = \sigma(r,s)\sum_{j=0}^{\infty}(j+1)^{-d(r)}(j+h+1)^{-d(s)} \tag{3.4}$$

for $h \geq 0$ and $r, s \in \mathbb{S}$. Observe that $\gamma_h(r,s) \geq 0$ for each $h \geq 0$ and each $r, s \in \mathbb{S}$. Let us denote

$$c(r,s) = \int_0^{\infty} x^{-d(r)}(x+1)^{-d(s)}\mathrm{d}x, \qquad c(s) = c(s,s) \tag{3.5}$$

and

$$d(r,s) = d(r) + d(s) \tag{3.6}$$

for $r, s \in \mathbb{S}$ provided that $1/2 < d(r) < 1$, $d(s) > 1/2$. Let us observe that $c(r,s) = \mathrm{B}(1-d(r), d(r,s)-1)$, where B is the beta function. $c(s)$ can be estimated from above with the following inequality

$$c(s) \leq \frac{1}{1-d(s)} + \frac{1}{2d(s)-1}. \tag{3.7}$$

Proposition 3.1 gives the asymptotic behaviour of $\gamma_h(r,s)$ and Proposition 3.2 provides a necessary and sufficient condition for the summability of the series $\sum_{k=0}^{\infty}\gamma_k(r,s)$. The notation $a_n \sim b_n$ indicates that the ratio of the two sequences tends to 1 as $n \to \infty$.

Proposition 3.1. *If $1/2 < d(r) < 1$ and $d(s) > 1/2$, then*

$$\gamma_h(r,s) \sim c(r,s)\sigma(r,s)\cdot h^{1-d(r,s)} \quad \textit{as} \quad h \to \infty,$$

where $c(r,s)$ is given by (3.5) *and $d(r,s)$ is given by* (3.6)*. If $d(r) = d(s) = 1$, then*

$$\gamma_h(r,s) \sim \sigma(r,s)\cdot h^{-1}\log h \quad \textit{as} \quad h \to \infty.$$

Proof. We approximate series (3.4) by integrals to obtain the following inequalities: if $1/2 < d(r) < 1$ and $d(s) > 1/2$, then we obtain

$$\sum_{j=0}^{\infty}(j+1)^{-d(r)}(j+h+1)^{-d(s)} \geq h^{1-d(r,s)}\int_{\frac{1}{h}}^{\infty} x^{-d(r)}(x+1)^{-d(s)}\mathrm{d}x, \tag{3.8}$$

$$\sum_{j=0}^{\infty}(j+1)^{-d(r)}(j+h+1)^{-d(s)} \leq h^{1-d(r,s)}\int_{0}^{\infty} x^{-d(r)}(x+1)^{-d(s)}\,\mathrm{d}x;$$

if $d(r) = d(s) = 1$, then we have that

$$\sum_{j=0}^{\infty}[(j+1)(j+h+1)]^{-1} \geq h^{-1}\Big[\log\Big(\frac{h+1}{2}\Big) + \int_1^{\infty}[y(y+1)]^{-1}\,\mathrm{d}y\Big]$$
$$\sum_{j=0}^{\infty}[(j+1)(j+h+1)]^{-1} \leq (h+1)^{-1} + h^{-1}\Big[\log\Big(\frac{h+1}{2}\Big) + \int_1^{\infty}[y(y+1)]^{-1}\,\mathrm{d}y\Big].$$

The proof is complete. □

Proposition 3.2. *The series*

$$\sum_{h=0}^{\infty}\gamma_h(r,s) \tag{3.9}$$

converges if and only if $d(s) > 1$ and $d(r,s) > 2$.

Proof. Series (3.9) has the following expression

$$\sum_{h=0}^{\infty}\gamma_k(r,s) = \sigma(r,s)\Big[\sum_{h=0}^{\infty}(h+1)^{-d(s)} + \sum_{h=0}^{\infty}\sum_{j=1}^{\infty}(j+1)^{-d(r)}(j+h+1)^{-d(s)}\Big].$$

The first series of the right-hand side of the equation above converges if and only if $d(s) > 1$. Thus, we only need to investigate the convergence of the series

$$\sum_{h=0}^{\infty}\sum_{j=1}^{\infty}(j+1)^{-d(r)}(j+h+1)^{-d(s)}. \tag{3.10}$$

A slight modification of inequality (3.8) shows that series (3.10) diverges if $d(r,s) \leq 2$. By choosing $\delta > 0$ such that $1 < 1+\delta < d(s)$ and $d(r,s) - \delta > 2$, we obtain

$$\begin{aligned}\sum_{h=0}^{\infty}\sum_{j=1}^{\infty}(j+1)^{-d(r)}(j+h+1)^{-d(s)} &= \sum_{h=0}^{\infty}\sum_{j=1}^{\infty}(j+1)^{-d(r)}(j+h+1)^{-d(s)+(1+\delta)-(1+\delta)} \\ &\leq \sum_{h=0}^{\infty}(h+1)^{-(1+\delta)}\sum_{j=1}^{\infty}(j+1)^{-d(r,s)+(1+\delta)}\end{aligned}$$

so that series (3.10) converges if $d(s) > 1$ and $d(r,s) > 2$. □

Remark 3.1. The series $\sum_{k=0}^{\infty}\gamma_k(s)$ converges if and only if $d(s) > 1$.

Let $\mathcal{L}_2(\mu) = \mathcal{L}_2(\mathbb{S}, \mathcal{S}, \mu)$ be a separable space of real-valued square μ-integrable functions with a seminorm

$$\|f\| = \Big[\int_{\mathbb{S}}|f(v)|^2\mu(\mathrm{d}v)\Big]^{1/2}, \quad f \in \mathcal{L}_2(\mu),$$

and let $L_2(\mu) = L_2(\mathbb{S}, \mathcal{S}, \mu)$ be the corresponding Hilbert space of equivalence classes of μ-almost everywhere equal functions with an inner product

$$\langle f, g\rangle = \int_{\mathbb{S}} f(v)g(v)\mu(\mathrm{d}v), \quad f, g \in L_2(\mu).$$

With an abuse of notation, we denote by f both a function and its equivalence class to avoid cumbersome notation. The intended meaning should be clear from the context.

Proposition 3.3 establishes a necessary and sufficient condition for the sample paths of the stochastic process $\{X_k(s) : s \in \mathbb{S}\}$ to be almost surely square μ-integrable with $\mathrm{E}\,\|X_k\|^2 < \infty$ for each $k \geq 1$.

Proposition 3.3. *The sample paths of the stochastic process $\{X_k(s) : s \in \mathbb{S}\}$ almost surely belong to the space $\mathcal{L}_2(\mu)$ and $\mathrm{E}\,\|X_k\|^2 < \infty$ for each $k \in \mathbb{Z}$ if and only if both of the integrals*

$$\mathrm{E}\,\|\varepsilon_0\|^2 = \int_{\mathbb{S}} \sigma^2(v)\mu(\mathrm{d}v) \qquad \textit{and} \qquad \int_{\mathbb{S}} \frac{\sigma^2(v)}{2d(v)-1}\mu(\mathrm{d}v) \tag{3.11}$$

are finite.

Proof. We show that the expected value

$$\mathrm{E}\Big[\int_{\mathbb{S}} X_0^2(v)\mu(\mathrm{d}v)\Big]$$

is finite if and only if integrals (3.11) are finite. First, using Fubini's theorem we obtain

$$\mathrm{E}\Big[\int_{\mathbb{S}} X_0^2(v)\mu(\mathrm{d}v)\Big] = \int_{\mathbb{S}} \mathrm{E}\,X_0^2(v)\mu(\mathrm{d}v).$$

Secondly, setting $h = 0$ and $r = s$ in equation (3.4) gives the expression for the variance

$$\mathrm{E}\,X_0^2(s) = \sigma^2(s)\sum_{j=0}^{\infty}(j+1)^{-2d(s)}$$

for $s \in \mathbb{S}$. Approximation of the series above by integrals leads to the following inequalities

$$2\int_{S} \mathrm{E}\,X_0^2(v)\mu(\mathrm{d}v) \geq \int_{\mathbb{S}} \sigma^2(v)\mu\mathrm{d}v + \int_{\mathbb{S}} \frac{\sigma^2(v)}{2d(v)-1}\mu(\mathrm{d}v),$$
$$\int_{\mathbb{S}} \mathrm{E}\,X_0^2(v)\mu(\mathrm{d}v) \leq \int_{\mathbb{S}} \sigma^2(v)\mu(\mathrm{d}v) + \int_{\mathbb{S}} \frac{\sigma^2(v)}{2d(v)-1}\mu(\mathrm{d}v).$$

The proof is complete. □

A stochastic process $\{\xi(s) : s \in \mathbb{S}\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with sample paths in $\mathcal{L}_2(\mu)$ induces the $\mathcal{F} - \mathcal{B}(L_2(\mu))$-measurable function $\omega \to \{\xi(s)(\omega) : s \in \mathbb{S}\} : \Omega \to L_2(\mu)$, where $\mathcal{B}(L_2(\mu))$ is the Borel σ-algebra of $L_2(\mu)$ (for more details, see Cremers and Kadelka [11]). Therefore we shall frequently consider each stochastic process $\{\xi(s) : s \in \mathbb{S}\}$ with sample paths in $\mathcal{L}_2(\mu)$ as a random element with values in $L_2(\mu)$ and denote it by $\{\xi(s) : s \in \mathbb{S}\}$ or simply by ξ.

Second approach

Now we establish a necessary and sufficient condition for the mean square convergence of series (3.1) with $\{a_j\}$ given by (3.2). Recall that $(j+1)^{-D} f = \{(j+1)^{-d(s)} f(s) : s \in \mathbb{S}\}$ for each $j \geq 0$ and $f \in L_2(\mu)$ since $e^T = \sum_{j=0}^{\infty} T^j/j!$ and $\lambda^T = e^{T \log \lambda}$ for $T \in L(\mathbb{E})$ and $\lambda > 0$.

Proposition 3.4. *Series* (3.1) *with a_j given by* (3.2) *and independent and identically distributed $L_2(\mu)$-valued random elements $\{\varepsilon_k\}$ such that* $\mathrm{E}\,\varepsilon_0 = 0$ *and* $\mathrm{E}\,\|\varepsilon_0\|^2 < \infty$ *converges in mean square if and only if there exists a measurable set $\mathbb{S}_0 \subset \mathbb{S}$ such that $\mu(\mathbb{S} \setminus \mathbb{S}_0) = 0$, $d(s) > 1/2$ for all $s \in \mathbb{S}_0$ and the integral*

$$\int_{\mathbb{S}} \frac{\sigma^2(v)}{2d(v) - 1} \mu(\mathrm{d}v)$$

is finite.

Proof. Let $N > M$, $\sigma^2(s) = \mathrm{E}\,\varepsilon_0^2(s)$ for $s \in \mathbb{S}$ and observe that

$$\mathrm{E}\Bigg\| \sum_{j=M+1}^{N} (j+1)^{-D} \varepsilon_{j-k} \Bigg\|^2 = \sum_{j=M+1}^{N} \int_{\mathbb{S}} (j+1)^{-2d(v)} \sigma^2(v) \mu(\mathrm{d}v).$$

Since

$$\sum_{j=0}^{\infty} \int_{\mathbb{S}} (j+1)^{-2d(v)} \sigma^2(v) \mu(\mathrm{d}v) = \int_{\mathbb{S}} \sum_{j=1}^{\infty} j^{-2d(v)} \sigma^2(v) \mu(\mathrm{d}v)$$

and

$$\frac{1}{2d(v) - 1} \leq \sum_{j=1}^{\infty} j^{-2d(v)} \leq 1 + \frac{1}{2d(v) - 1}$$

we have that

$$\int_{\mathbb{S}} \frac{\sigma^2(v)}{2d(v) - 1} \mu(\mathrm{d}v) \leq \int_{\mathbb{S}} \sigma^2(v) \sum_{j=1}^{\infty} j^{-2d(v)} \mu(\mathrm{d}v) \leq \mathrm{E}\,\|\varepsilon_0\|^2 + \int_{\mathbb{S}} \frac{\sigma^2(v)}{2dv) - 1} \mu(\mathrm{d}v)$$

and the proof is complete. □

Remark 3.2. Since $\{\varepsilon_k\}$ are independent, it follows from Lévy-Itô-Nisio theorem (see Ledoux and Talagrand [35], Theorem 6.1, p. 151) and Proposition 3.4 that series (3.1) also converges almost surely. Hence, X_k for each $k \in \mathbb{Z}$ is an $L_2(\mu)$-valued random element and Proposition 3.4 is consistent with Proposition 3.3.

Remark 3.3. Since $\{a_j\}$ given by (3.2) are multiplication operators from $L_2(\mu)$ to $L_2(\mu)$, we have that the operator norm $\|(j+1)^{-D}\| = \inf\{c > 0 : \mu(s \in \mathbb{S} : |(j+1)^{-d(s)}| > c) = 0\}$ (see Theorem 1.5 of Conway [9]). If $\underline{d} = \operatorname{ess\,inf} d = 1/2$, then we have that $\sum_{j=0}^{\infty} \|u_j\|^2 = \sum_{j=1}^{\infty} j^{-1} = \infty$, but series (3.1) might still converge. The square summability of the operator norms of $\{a_j\}$ is not a necessary condition for the almost sure convergence of series (3.1).

3.1.2 Asymptotic behaviour of cross-covariances

$\zeta_n(t)$ can be expressed as a series

$$\zeta_n(t) = \sum_{j=-\infty}^{\lfloor nt \rfloor + 1} a_{nj}(t)\varepsilon_j$$

for each $t \in [0,1]$, where

$$a_{nj}(t) = \sum_{k=1}^{\lfloor nt \rfloor} v_{k-j} + \{nt\} v_{\lfloor nt \rfloor + 1 - j} \tag{3.12}$$

and

$$v_j = \begin{cases} u_j, & \text{if } j \geq 0; \\ 0, & \text{if } j < 0. \end{cases} \tag{3.13}$$

We adopt the usual convention that an empty sum equals 0.

Denote

$$S_n(s) = \sum_{k=1}^{n} X_k(s)$$

for each $s \in \mathbb{S}$ and

$$\zeta_n(s,t) = S_{\lfloor nt \rfloor}(s) + \{nt\} X_{\lfloor nt \rfloor + 1}(s)$$

for each $s \in \mathbb{S}$ and each $t \in [0,1]$. Each random variable $\zeta_n(s,t)$ can be expressed as a series

$$\zeta_n(s,t) = \sum_{j=-\infty}^{\lfloor nt \rfloor + 1} a_{nj}(s,t)\varepsilon_j(s),$$

where

$$a_{nj}(s,t) = \sum_{k=1}^{\lfloor nt \rfloor} v_{k-j}(s) + \{nt\} v_{\lfloor nt \rfloor + 1 - j}(s)$$

and

$$v_j(s) = \begin{cases} (j+1)^{-d(s)}, & \text{if } j \geq 0; \\ 0, & \text{if } j < 0. \end{cases} \tag{3.14}$$

Observe that $v_j = v_j(s)$ if $d(s) = 1$ for each $s \in \mathbb{S}$ since $u_j = (j+1)^{-1}$ if $d(s) = 1$ for each $s \in \mathbb{S}$. Notice that the upper bounds of summation of the series in the expressions of $\zeta_n(t)$ and $\zeta_n(s,t)$ can be extended up to ∞ since $a_{nj}(s,t) = 0$ and $a_{nj}(t) = 0$ if $j > \lfloor nt \rfloor + 1$.

The growth rate of the cross-covariance of the partial sums of the sequences $\{X_k(s)\}$ and $\{X_k(t)\}$ for $s, t \in \mathbb{S}$ is established in Proposition 3.5.

Proposition 3.5. *If* $1/2 < d(s) < 1$ *and* $1/2 < d(t) < 1$*, then*

$$\mathrm{E}\big[S_n(r)S_n(s)\big] \sim \frac{[c(r,s)+c(s,r)]\sigma(r,s)}{[2-d(r,s)][3-d(r,s)]} \cdot n^{3-d(r,s)} \quad as \quad n \to \infty, \tag{3.15}$$

where $c(r,s)$ *is given by* (3.5) *and* $d(r,s)$ *is given by* (3.6)*.*

If $d(r) = d(s) = 1$*, then*

$$\mathrm{E}\big[S_n(r)S_n(s)\big] \sim \sigma(r,s) \cdot n \log^2 n \quad as \quad n \to \infty. \tag{3.16}$$

Proof. The cross-covariance of the partial sums of the sequences $\{X_k(s)\}$ and $\{X_k(t)\}$ has the following expression

$$\begin{aligned} \mathrm{E}[S_n(r)S_n(s)] &= n\gamma_0(r,s) \\ &\quad + \sum_{k=1}^{n-1}\sum_{l=k+1}^{n} \mathrm{E}[X_k(r)X_l(s)] + \sum_{k=1}^{n-1}\sum_{l=k+1}^{n} \mathrm{E}[X_k(s)X_l(r)]. \end{aligned} \tag{3.17}$$

Since

$$\sum_{k=1}^{n-1}\sum_{l=k+1}^{n} \mathrm{E}[X_k(r)X_l(s)] = n\sum_{k=1}^{n-1}\gamma_k(r,s) - \sum_{k=1}^{n-1} k\gamma_k(r,s),$$

we can use the results of Proposition 3.1 to obtain the following asymptotic relations: if $1/2 < d(r) < 1$ and $1/2 < d(s) < 1$, then

$$\sum_{k=1}^{n-1}\gamma_k(r,s) \sim \frac{c(r,s)\sigma(r,s)}{2-d(r,s)} \cdot n^{2-d(r,s)} \quad \text{and} \quad \sum_{k=1}^{n-1} k\gamma_k(r,s) \sim \frac{c(r,s)\sigma(r,s)}{3-d(r,s)} \cdot n^{3-d(r,s)}$$

as $n \to \infty$; if $d(s) = 1$ and $d(t) = 1$, then

$$\sum_{k=1}^{n-1} \gamma_k(r,s) \sim \frac{\sigma(r,s)}{2} \cdot \ln^2 n \quad \text{and} \quad \sum_{k=1}^{n-1} k\gamma_k(r,s) \sim \sigma(r,s) \cdot n \ln n$$

as $n \to \infty$. $\square$

Remark 3.4. The asymptotic behaviour of the variance of the partial sums of the sequence $\{X_k(s)\}$ is the following: if $1/2 < d(s) < 1$, then

$$\mathrm{E}\, S_n^2(s) \sim \frac{c(s)\sigma^2(s)}{[1-d(s)][3-2d(s)]} \cdot n^{3-2d(s)};$$

if $d(s) = 1$, then

$$\mathrm{E}\, S_n^2(s) \sim \sigma^2(s) \cdot n \ln^2 n.$$

Set $\mathbb{T} = \mathbb{S} \times [0,\infty)$ and define the function $V : \mathbb{T}^2 \to \mathbb{R}$ by

$$V((r,t),(s,u)) = \frac{\sigma(r,s)}{[2-d(r,s)][3-d(r,s)]} \big[c(s,r)t^{3-d(r,s)} + c(r,s)u^{3-d(r,s)} - C(r,s;t-u)|t-u|^{3-d(r,s)}\big], \quad (3.18)$$

where $d(r,s)$ is given by (3.6), $c(r,s)$ is given by (3.5) and

$$C(r,s;t) = \begin{cases} c(r,s) & \text{if } t < 0; \\ c(s,r) & \text{if } t > 0. \end{cases}$$

Now we are prepared to derive the asymptotic behavior of the sequence of cross-covariances of ζ_n.

Proposition 3.6. *Suppose either* $1/2 < d(r) < 1$ *and* $1/2 < d(s) < 1$ *or* $d(r) = d(s) = 1$. *In both cases, the following asymptotic relation holds*

$$\mathrm{E}[\zeta_n(r,t)\zeta_n(s,u)] \sim \mathrm{E}[S_{\lfloor nt \rfloor}(r) S_{\lfloor nu \rfloor}(s)].$$

Proposition 3.7. *If* $1/2 < d(r) < 1$ *and* $1/2 < d(s) < 1$, *then*

$$\mathrm{E}[S_{\lfloor nt \rfloor}(r) S_{\lfloor nu \rfloor}(s)] \sim V((r,t),(s,u)) \cdot n^{3-d(r,s)}$$

for $(r,t),(s,u) \in \mathbb{S} \times [0,1]$, *where* V *is given by* (3.18).

If $d(r) = d(s) = 1$, *then*

$$\mathrm{E}[S_{\lfloor nt \rfloor}(r) S_{\lfloor nu \rfloor}(s)] \sim \sigma(r,s) \cdot \min(t,u) \cdot n \log^2 n.$$

Remark 3.5. Let us assume that $r = s$ and $1/2 < d(s) < 1$. By setting $r = s$ in Proposition 3.7 and using Proposition 3.6, we obtain that

$$\mathrm{E}[\zeta_n(s,t)\zeta_n(s,u)] \sim \frac{\sigma^2(s)c(s)}{[1-d(s)][3-2d(s)]} \cdot \mathrm{E}[B_{3/2-d(s)}(t)B_{3/2-d(s)}(u)] \cdot n^{3-2d(s)},$$

where

$$\mathrm{E}[B_{3/2-d(s)}(t)B_{3/2-d(s)}(u)] = \frac{1}{2}[t^{3-2d(s)} + u^{3-2d(s)} - |t-u|^{3-2d(s)}]$$

is the covariance function of the fractional Brownian motion

$$B_{3/2-d(s)} = \{B_{3/2-d(s)}(t) : t \in [0,1]\}$$

with the Hurst parameter $3/2 - d(s)$ and $c(s)$ is given by (3.5).

Remark 3.6. The asymptotic behaviour of the variance $\mathrm{E}\,\zeta_n^2(s,t)$ follows from Proposition 3.6 and Proposition 3.7 by setting $r = s$ and $t = u$: if $1/2 < d(s) < 1$, then

$$\mathrm{E}\,\zeta_n^2(s,t) \sim \frac{c(s)\sigma^2(s)}{[1-d(s)][3-2d(s)]} \cdot t^{3-2d(s)} \cdot n^{3-2d(s)};$$

if $d(s) = 1$, then

$$\mathrm{E}\,\zeta_n^2(s,t) \sim \sigma^2(s) \cdot t \cdot n \log^2 n.$$

Proof of Proposition 3.7. Suppose $t < u$ and split the cross-covariance of the partial sums into two terms

$$\mathrm{E}\big[S_{\lfloor nt \rfloor}(r)S_{\lfloor nu \rfloor}(s)\big] = \mathrm{E}\big[S_{\lfloor nt \rfloor}(r)S_{\lfloor nt \rfloor}(s)\big] + \mathrm{E}\big[S_{\lfloor nt \rfloor}(r)\big[S_{\lfloor nu \rfloor}(s) - S_{\lfloor nt \rfloor}(s)\big]\big]. \quad (3.19)$$

The asymptotic behaviour of the first term of sum (3.19) is established using (3.15) and (3.16): if $1/2 < d(r) < 1$ and $1/2 < d(s) < 1$, then

$$\mathrm{E}[S_{\lfloor nt \rfloor}(r)S_{\lfloor nt \rfloor}(s)] \sim \frac{[c(r,s)+c(s,r)]\sigma(r,s)}{[2-d(r,s)][3-d(r,s)]} \cdot t^{3-d(r,s)} \cdot n^{3-d(r,s)}; \quad (3.20)$$

if $d(r) = d(s) = 1$, then

$$\mathrm{E}[S_{\lfloor nt \rfloor}(r)S_{\lfloor nt \rfloor}(s)] \sim \sigma(r,s) \cdot t \cdot n \log^2 n. \quad (3.21)$$

In order to establish the asymptotic behaviour of the second term of sum (3.19), we express it in the following way

$$\mathrm{E}\big[S_{\lfloor nt\rfloor}(r)[S_{\lfloor nu\rfloor}(s) - S_{\lfloor nt\rfloor}(s)]\big] = \sum_{k=1}^{m_n-1} k[\gamma_k(r,s) + \gamma_{\lfloor nu\rfloor-k}(r,s)] + m_n \sum_{k=0}^{|\lfloor nu\rfloor-2\lfloor nt\rfloor|} \gamma_{m_n+k}(r,s), \quad (3.22)$$

where $m_n := \min(\lfloor nt\rfloor, \lfloor nu\rfloor - \lfloor nt\rfloor)$ (we also use the notation $m := \min(t, u-t)$). For simplicity, denote

$$\kappa(a,b) = \sum_{k=a+1}^{b} \gamma_k(r,s) \quad \text{and} \quad \nu(a,b) = \sum_{k=a+1}^{b} k\gamma_k(r,s).$$

Then we have that

$$\sum_{k=1}^{m_n-1} k\gamma_{\lfloor nu\rfloor-k}(r,s) = \lfloor nu\rfloor\kappa(\lfloor nu\rfloor-m_n, \lfloor nu\rfloor-1) - \nu(\lfloor nu\rfloor-m_n, \lfloor nu\rfloor-1). \quad (3.23)$$

Let us a recall a few facts about sequences. We use these facts to establish asymptotic behaviour of the sums in (3.22) and (3.23). Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of positive real numbers such that $a_n \sim b_n$. Then $\sum_{k=1}^n a_k \sim \sum_{k=1}^n b_k$ provided either of these partial sums diverges. Let f be a continuous strictly increasing or strictly decreasing function such that $f(x)/f(x+1) \to 1$ as $x \to \infty$ and $\int_1^n f(x)\mathrm{d}x \to \infty$ as $n \to \infty$. Then $\sum_{k=1}^n f(k) \sim \int_1^n f(x)\mathrm{d}x$.

Since $\gamma_k(r,s) \sim c(r,s)\sigma(r,s) \cdot k^{1-d(r,s)}$ if $1/2 < d(r) < 1$ and $d(s) > 1/2$ (see Proposition 3.1), we obtain the following asymptotic relations using the facts about sequences mentioned above:

$$\nu(0, m_n - 1) \sim \frac{c(r,s)\sigma(r,s)m^{3-d(r,s)}}{3-d(r,s)} \cdot n^{3-d(r,s)} \quad (3.24)$$

$$\lfloor nu\rfloor\kappa(\lfloor nu\rfloor - m_n, \lfloor nu\rfloor - 1) \sim \sim \frac{c(r,s)\sigma(r,s)u[u^{2-d(r,s)} - (u-m)^{2-d(r,s)}]}{2-d(r,s)} \cdot n^{3-d(r,s)}; \quad (3.25)$$

$$\nu(\lfloor nu\rfloor - m_n, \lfloor nu\rfloor - 1) \sim \sim \frac{c(r,s)\sigma(r,s)[u^{3-d(r,s)} - (u-m)^{3-d(r,s)}]}{3-d(r,s)} \cdot n^{3-d(r,s)}; \quad (3.26)$$

$$m_n\kappa(m_n - 1, m_n + |\lfloor nu\rfloor - 2\lfloor nt\rfloor|) \sim \sim \frac{c(r,s)\sigma(r,s)m[(m+|u-2t|)^{2-d(r,s)} - m^{2-d(r,s)}]}{2-d(r,s)} \cdot n^{3-d(r,s)}. \quad (3.27)$$

We have that

$$\mathrm{E}\big[S_{\lfloor nt\rfloor}(r)[S_{\lfloor nu\rfloor}(s)-S_{\lfloor nt\rfloor}(s)]\big] \sim$$
$$\sim \frac{c(r,s)\sigma(r,s)}{[2-d(r,s)][3-d(r,s)]}\big[-t^{3-d(r,s)}+u^{3-d(r,s)}-(u-t)^{3-d(r,s)}\big]\cdot n^{3-d(r,s)} \quad (3.28)$$

using asymptotic relations (3.24)-(3.27). Combining (3.20) with (3.28), we obtain

$$\mathrm{E}[S_{\lfloor nt\rfloor}(r)S_{\lfloor nu\rfloor}(s)] \sim \frac{\sigma(r,s)}{[2-d(r,s)][3-d(r,s)]}\big[c(s,r)t^{3-d(r,s)} + c(r,s)[u^{3-d(r,s)}-(u-t)^{3-d(r,s)}]\big]\cdot n^{3-d(r,s)}.$$

Similarly, if $d(r)=d(s)=1$, then $\gamma_k(r,s)\sim\sigma(r,s)\cdot k^{-1}\log k$ (see Proposition 3.1) and the following asymptotic relations are true

$$\nu(0,m_n-1)\sim\sigma(r,s)m\cdot n\log n; \quad (3.29)$$

$$\lfloor nu\rfloor\kappa(\lfloor nu\rfloor-m_n,\lfloor nu\rfloor-1)\sim\sigma(r,s)[\log u-\log(u-m)]u\cdot n\log n; \quad (3.30)$$

$$\nu(\lfloor nu\rfloor-m_n,\lfloor nu\rfloor-1)\sim\sigma(r,s)m\cdot n\log n; \quad (3.31)$$

$$m_n\kappa(m_n-1,m_n+|\lfloor nu\rfloor-2\lfloor nt\rfloor|)\sim$$
$$\sim\sigma(r,s)[\log(m+|u-2t|)-\log m]m\cdot n\log n. \quad (3.32)$$

Since sequences (3.29)-(3.32) grow slower than sequence (3.21), we conclude that

$$\mathrm{E}\big[S_{\lfloor nt\rfloor}(r)S_{\lfloor nu\rfloor}(s)\big]\sim\sigma(r,s)\cdot t\cdot n\log^2 n.$$

If $t>u$, the proof is exactly the same as in the case of $t<u$. If $t=u$, then we just use asymptotic relations (3.20) and (3.21). The proof of Proposition 3.7 is complete. □

Proof of Proposition 3.6. We have that

$$\begin{aligned}\mathrm{E}[\zeta_n(r,t)\zeta_n(s,u)] = {} & \mathrm{E}[S_{\lfloor nt\rfloor}(r)S_{\lfloor nu\rfloor}(s)] \\ & + \{nu\}\,\mathrm{E}[S_{\lfloor nt\rfloor}(r)X_{\lfloor nu\rfloor+1}(s)] \\ & + \{nt\}\,\mathrm{E}[S_{\lfloor nu\rfloor}(s)X_{\lfloor nt\rfloor+1}(r)] \\ & + \{nt\}\{nu\}\,\mathrm{E}[X_{\lfloor nt\rfloor+1}(r)X_{\lfloor nu\rfloor+1}(s)]\end{aligned}$$

and

$$\mathrm{E}[S_{\lfloor nt\rfloor}(r)X_{\lfloor nu\rfloor+1}(s)]\le\lfloor nt\rfloor\gamma_0(r,s).$$

The result follows from Proposition 3.7 since $\mathrm{E}[S_{\lfloor nt\rfloor}(r)S_{\lfloor nu\rfloor}(s)]$ is the only term in the expression of $\mathrm{E}[\zeta_n(r,t)\zeta_n(s,u)]$ that grows faster than linearly. □

3.1.3 Operator self-similar process

In this section, we show that there exists a Gaussian stochastic process $\mathcal{X} = \{\mathcal{X}(s,t) : (s,t) \in \mathbb{T}\}$ with zero mean and covariance function V given by (3.18). The stochastic process $\{\mathcal{X}(\cdot,t) : t \in [0,\infty)\}$ is an operator self-similar process with values in $L_2(\mu)$.

We begin by showing that the function V is a covariance function.

Proposition 3.8. *The function* $V : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$*, given by* (3.18)*, with* $d \in (1/2, 1)$ *is a covariance function of a stochastic process indexed by the set* $\mathbb{T}$*.*

Proof. It follows from equation (3.18) that the function V is symmetric, i.e.

$$V(\tau,\tau') = V(\tau',\tau), \quad \tau,\tau' \in \mathbb{T}.$$

So we need to prove that the function V is positive definite. Let $N \in \mathbb{N}$, $\tau_i = (s_i,t_i) \in \mathbb{T}$ and $w_i \in \mathbb{R}$, where $i \in \{1,\dots,N\}$. Denote $M = \max\{t_1,\dots,t_N\}$ and $\widetilde{w}_i = w_i M^{3/2-d(s_i)}$, $i \in \{1,\dots,N\}$. Using equation (3.18) and Propositions 3.6 and 3.7, we obtain that

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j V(\tau_i,\tau_j) = \\
&\quad = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j M^{3-[d(s_i)+d(s_j)]} V((s_i,t_i/M),(s_j,t_j/M)) \\
&\quad = \sum_{i=1}^{N}\sum_{j=1}^{N} \widetilde{w}_i\widetilde{w}_j \lim_{n\to\infty} \frac{1}{n^{3-[d(s_i)+d(s_j)]}} \mathrm{E}[\zeta_n(s_i,t_i/M)\zeta_n(s_j,t_j/M)] \geq 0
\end{aligned}$$

since

$$\frac{1}{n^{3-d(r,s)}} \mathrm{E}[\zeta_n(r,t)\zeta_n(s,u)]$$

is a covariance function for all $(r,t),(s,u) \in \mathbb{S}\times[0,1]$ and for all $n \in \mathbb{N}$. $\square$

Let us recall that a random element ξ with values in a separable Banach space $\mathbb{E}$ is Gaussian if for any continuous linear functional f on $\mathbb{E}$, $f(\xi)$ is real valued Gaussian random variable. A stochastic process $\{\xi_t : t \in T\}$ with values in $\mathbb{E}$ is Gaussian if each finite linear combination $\sum_i \alpha_i\xi_{t_i}$, $\alpha_i \in \mathbb{R}$, $t_i \in T$, is Gaussian random element in $\mathbb{E}$ (for more details about Gaussian random elements and

Gaussian stochastic processes with values in Banach spaces, see the textbook by Ledoux and Talagrand [35]).

We have the following corollary of Proposition 3.8.

Corollary. *There exists a zero mean Gaussian stochastic process* $\mathcal{X} = \{\mathcal{X}(s,t) : (s,t) \in \mathbb{T}\}$ *with the covariance function* V *given by* (3.18).

Next we describe the sample path properties of the stochastic process $\mathcal{X}$. First we consider for each $t \in [0,\infty)$ the stochastic process $\{\mathcal{X}(s,t) : s \in \mathbb{S}\}$.

Proposition 3.9. *If* $d \in (1/2, 1)$ *and the integrals*

$$\int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)]^2}\mu(\mathrm{d}v) \quad \textit{and} \quad \int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)][2d(v)-1]}\mu(\mathrm{d}v)$$

are finite, then for each $t \in [0,\infty)$ *the stochastic process* $\{\mathcal{X}(s,t) : s \in \mathbb{S}\}$ *has sample paths in* $\mathcal{L}_2(\mu)$ *and induces a Gaussian random element with values in* $L_2(\mu)$ *which is denoted by* $\mathcal{X}(\cdot,t)$. *Moreover, the process* $\{\mathcal{X}(\cdot,t) : t \in [0,\infty)\}$ *with values in* $L_2(\mu)$ *is Gaussian.*

Proof. Since we have that

$$\begin{aligned}
\mathrm{E}\int_{\mathbb{S}} \mathcal{X}^2(v,t)\mu(\mathrm{d}v) &= \int_{\mathbb{S}} \frac{\sigma^2(v)c(v)}{[1-d(v)][3-2d(v)]} \cdot t^{3-2d(v)}\mu(\mathrm{d}v) \\
&\le \max\{t,t^2\}\Big[\int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)]^2}\mu(\mathrm{d}v) \\
&\qquad + \int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)][2d(v)-1]}\mu(\mathrm{d}v)\Big]
\end{aligned}$$

using inequality (3.7) to estimate $c(s)$ from above, the sample paths of the stochastic process $\{\mathcal{X}(s,t) : s \in \mathbb{S}\}$ almost surely belong to the space $\mathcal{L}_2(\mu)$ for each $t \in [0,\infty)$. Hence $\mathcal{X}(\cdot,t)$ is a random element in $L_2(\mu)$. Clearly it is a Gaussian one. □

Finally, we show that the stochastic process $\{\mathcal{X}(\cdot,t) : t \in [0,\infty)\}$ is operator self-similar.

Proposition 3.10. *The stochastic process* $\{\mathcal{X}(\cdot,t) : t \in [0,\infty)\}$ *is operator self-similar with scaling family of operators* $\{a^H : a > 0\}$ *where* a^H, $a > 0$, *is a multiplication operator defined by* $a^H f = \{a^{3/2-d(s)}f(s) : s \in \mathbb{S}\}$ *for* $f \in L_2(\mu)$.

Proof. We need to show that

$$\{\mathcal{X}(\cdot, at) : t \in [0, \infty)\} \overset{fdd}{=} \{a^H \mathcal{X}(\cdot, t) : t \in [0, \infty)\}. \tag{3.33}$$

Since stochastic processes on both sides of equality (3.33) are zero-mean Gaussian stochastic processes, we only need to show that their covariance structure is the same. Using the fact that two operators A and B are equal if and only if $\langle Af, g\rangle = \langle Bf, g\rangle$ for all $f, g \in L_2(\mu)$ and the fact that

$$\mathrm{E}[a^{3/2-d(r)}\mathcal{X}(r,t)a^{3/2-d(s)}\mathcal{X}(s,u)] = E[\mathcal{X}(r,at)\mathcal{X}(s,au)] \tag{3.34}$$

for all $r, s \in \mathbb{S}$ and $t, u \in [0, \infty)$ (equality (3.34) follows from equation (3.18)), we conclude the proof by showing that

$$\begin{aligned}
&\langle \mathrm{E}[\langle a^H \mathcal{X}(\cdot, t), f\rangle a^H \mathcal{X}(\cdot, u)], g\rangle = \\
&= \int_{\mathbb{S}} \mathrm{E}\left[\left(\int_{\mathbb{S}} a^{3/2-d(u)}\mathcal{X}(u,t)f(u)\mu(\mathrm{d}u)\right) a^{3/2-d(r)}\mathcal{X}(r,u)\right] g(r)\mu(\mathrm{d}r) \\
&= \int_{\mathbb{S}}\left(\int_{\mathbb{S}} \mathrm{E}[a^{3/2-d(u)}\mathcal{X}(u,t)a^{3/2-d(r)}\mathcal{X}(r,u)]f(u)\mu(\mathrm{d}u)\right) g(r)\mu(\mathrm{d}r) \\
&= \langle \mathrm{E}[\langle \mathcal{X}(\cdot, at), f\rangle \mathcal{X}(\cdot, au)], g\rangle
\end{aligned}$$

for all $f, g \in L_2(\mu)$. □

3.1.4 Main tools

We use three auxiliary results in the proofs of the central limit theorem and the functional central limit theorem. The first result gives sufficient conditions for the convergence in distribution of random processes with sample paths in $\mathcal{L}_2(\mu; \mathbb{E})$. The second result is used to establish convergence in distribution random elements with specific structure and values in a separable Hilbert space. The last result gives sufficient conditions for tightness of random elements with values in $C([0, 1]; \mathbb{H})$, where $\mathbb{H}$ is a separable Hilbert space.

Convergence of random processes with sample paths in $\mathcal{L}_2(\mu; \mathbb{E})$

Let $\mathbb{E}$ be a separable Banach space with Borel σ-algebra $\mathcal{B}(\mathbb{E})$. Suppose that $\mathcal{L}_2(\mu; \mathbb{E}) = \mathcal{L}_2(\mathbb{S}, \mathcal{S}, \mu; \mathbb{E})$ is a separable space of square μ-integrable $\mathbb{E}$-valued functions with a seminorm

$$\|f\| = \left[\int_{\mathbb{S}} \|f(v)\|^2 \mu(\mathrm{d}v)\right]^{1/2}$$

for $f \in \mathcal{L}_2(\mu;\mathbb{E})$. Let $L_2(\mu;\mathbb{E}) = L_2(\mathbb{S},\mathcal{S},\mu;\mathbb{E})$ be the corresponding Banach space of equivalence classes of μ-almost everywhere equal functions.

Let $\{\xi_n\} = \{\xi_n : n \geq 0\}$ be a sequence of $\mathcal{F}\otimes\mathcal{S} - \mathcal{B}_E$-measurable functions $\xi_n : \Omega\times\mathcal{S} \to \mathbb{E}$ with $\xi_n(\omega,\cdot) \in \mathcal{L}_2(\mu;\mathbb{E})$ for all $\omega \in \Omega$ and for all $n \geq 0$, i.e. a sequence of measurable random process with values in $\mathbb{E}$ and sample paths in $\mathcal{L}_2(\mu;\mathbb{E})$. Then the maps $\hat{\xi}_n : \Omega \to L_2(\mu;\mathbb{E})$, $\omega \mapsto \hat{\xi}_n(\omega) = \xi_n(\omega,\cdot)$ for each $n \geq 0$ are $\mathcal{F} - \mathcal{B}(L_2(\mu;\mathbb{E}))$ measurable (see Cremers and Kadelka [11]) and the distributions of $\hat{\xi}_n$ are well-defined probability measures on $(L_2(\mu;\mathbb{E}),\mathcal{B}(L_2(\mu;\mathbb{E})))$. It is said that ξ_n converges in distribution to ξ_0 and written $\xi_n \xrightarrow{\mathcal{D}} \xi_0$ if and only if the distributions of $\hat{\xi}_n$ converge weakly to the distribution of $\hat{\xi}_0$, i.e. $\mathrm{E}\, f(\hat{\xi}_n) \to \mathrm{E}\, f(\hat{\xi}_0)$ for all bounded continuous functions $f : L_2(\mu;\mathbb{E}) \to \mathbb{R}$.

Now we are ready to state theorem which is proved by Cremers and Kadelka [12].

Theorem 3.1. *Let $\{\xi_n\}$ be a sequence of random processes with sample paths in $\mathcal{L}_2(\mu;\mathbb{E})$. Then $\xi_n \xrightarrow{\mathcal{D}} \xi_0$ as $n \to \infty$ provided that the following three conditions are satisfied:*

(I) the finite-dimensional distributions of ξ_n converge weakly to those of ξ_0 almost everywhere;

(II) (a) for each $s \in \mathbb{S}$, $\mathrm{E}\,\|\xi_n(s)\|^2 \to \mathrm{E}\,\|\xi_0(s)\|^2$ as $n \to \infty$;

(b) there exists a μ-integrable function $f : \mathbb{S} \to [0,\infty)$ such that

$$\mathrm{E}\,\|\xi_n(s)\|^2 \leq f(s)$$

for each $s \in \mathbb{S}$ and each $n \geq 1$.

Convergence in distribution of random elements with values in a separable Hilbert space

Let $\mathbb{E}$ and $\mathbb{F}$ be two separable Hilbert spaces and let $L(\mathbb{E},\mathbb{F})$ be the space of bounded linear operators from $\mathbb{E}$ to $\mathbb{F}$. Suppose that a sequence $\{Z_n\}$ of $\mathbb{F}$-valued random elements can be expressed as

$$Z_n = \sum_{j=-\infty}^{\infty} B_{nj}\xi_j,$$

where $\{B_{nj}\}$ is a sequence in $L(\mathbb{E},\mathbb{F})$ for each $n \in \mathbb{N}$ and $\{\xi_j\}$ is a sequence of independent and identically distributed $\mathbb{E}$-valued random elements with $\mathrm{E}\,\xi_0 = 0$ and $\mathrm{E}\,\|\xi_0\|^2 < \infty$. Using the same linear bounded operators $\{B_{nj}\}$, we construct another sequence $\{\tilde{Z}_n\}$ of $\mathbb{F}$-valued random elements that can be represented as

$$\tilde{Z}_n = \sum_{j=-\infty}^{\infty} B_{nj}\tilde{\xi}_j,$$

where $\{\tilde{\xi}_j\}$ is a sequence of independent and identically distributed $\mathbb{E}$-valued Gaussian random elements with $\mathrm{E}\,\tilde{\xi}_0 = 0$ and the same covariance operator as that of ξ_0.

Under the conditions of Lemma 3.1 below, the sequences $\{Z_n\}$ and $\{\tilde{Z}_n\}$ have the same limiting behaviour, i.e. if one converges in distribution then so does the other and their limits coincide. Before we state Lemma 3.1, we define the distance function ρ_k.

Definition 3.1. Let U and V be random elements with values in a separable Hilbert space $\mathbb{H}$. The distance function ρ_k is given by

$$\rho_k(U,V) = \sup_{f\in F_k} |\,\mathrm{E}\,f(U) - \mathrm{E}\,f(V)|,$$

where F_k is the set of k times Frechet differentiable functions $f : \mathbb{H} \to \mathbb{R}$ such that

$$\sup_{x\in\mathbb{H}} \left|f^{(j)}(x)\right| \le 1, \quad j = 0, 1, \ldots, k.$$

It is proved by Giné and León [18] that the distance function ρ_k metrizes the convergence in distribution of sequences of random elements with values in $\mathbb{H}$ for every $k > 0$.

Lemma 3.1. *If both of the conditions*

$$\lim_{n\to\infty} \sup_{j\in\mathbb{Z}} \|B_{nj}\| = 0 \tag{3.35}$$

and

$$\limsup_{n\to\infty} \sum_{j=-\infty}^{\infty} \|B_{nj}\|^2 < \infty \tag{3.36}$$

are satisfied, then $\lim_{n\to\infty} \rho_3(Z_n, \tilde{Z}_n) = 0$.

Proof. The proof follows from the proof of Proposition 4.1 of Račkauskas and Suquet [48]. The only difference is that $\mathbb{E} = \mathbb{F}$ in Račkauskas and Suquet [48], but the proof remains valid as long as

$$\|B_{nk}f\| \leq \|B_{nk}\|\|f\|$$

for each $n \in \mathbb{N}$, each $k \in \mathbb{Z}$ and each $f \in \mathbb{E}$. □

Tightness of random elements with values in $C([0,1];\mathbb{H})$

Suppose that $\{Z_n\} = \{Z_n : n \geq 1\}$ are random elements with values in $C([0,1] : \mathbb{H})$, where $\mathbb{H}$ is a separable Hilbert space. The following theorem is an adaptation of Theorem 12.3 of Billinglsey [3] (see also Proposition 4.2 of Račkauskas and Suquet [48]).

Proposition 3.11. *Let $\mathbb{H}$ be a separable Hilbert space. The sequence $\{Z_n\}$ of random elements of the space $C([0,1];\mathbb{H})$ is tight if*

(i) $\{Z_n(t)\}$ is tight on $\mathbb{H}$ for every $t \in [0,1]$;

(ii) there exists $\gamma \geq 0$, $a > 1$ and a continuous increasing function $F : [0,1] \to \mathbb{R}$ such that

$$P(\|Z_n(t) - Z_n(u)\| > \lambda) \leq \lambda^{-\gamma}|F(t) - F(u)|^a.$$

3.2 Central limit theorem

Suppose that $\{X_k\} = \{X_k : k \in \mathbb{Z}\}$ is an $L_2(\mu)$-valued linear process such that $\{a_j\}$ is given by $a_j = (j+1)^{-D}$ for each $j \geq 0$, where $D : L_2(\mu) \to L_2(\mu)$ is a multiplication operator defined by $Df = \{d(s)f(s) : s \in \mathbb{S}\}$ for each $f \in L_2(\mu)$ with a measurable function $d : \mathbb{S} \to \mathbb{R}$. Suppose that $\mathrm{E}\,\varepsilon_0 = 0$ and $\mathrm{E}\,\|\varepsilon_0\|^2 < \infty$.

Theorem 3.2. *Suppose that $1/2 < d(s) < 1$ for each $s \in \mathbb{S}$, $\mathrm{E}\,\varepsilon_0^2(s) < \infty$ for each $s \in \mathbb{S}$ and both of the integrals*

$$\int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)]^2}\mu(\mathrm{d}v) \quad \textit{and} \quad \int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)][2d(v)-1]}\mu(\mathrm{d}v) \tag{3.37}$$

are finite. Then

$$n^{-H}S_n \xrightarrow{\mathcal{D}} G \quad \textit{as} \quad n \to \infty,$$

where $\{n^{-H}\} = \{n^{-H} : n \geq 1\}$ *are multiplication operators given by* $n^{-H}f = \{n^{-[3/2-d(s)]}f(s) : s \in \mathbb{S}\}$ *for each* $f \in L^2(\mu)$, $G = \{G(s) : s \in \mathbb{S}\}$ *is a zero mean Gaussian random process with the autocovariance function*

$$\mathrm{E}[G(r)G(s)] = \frac{[c(r,s) + c(s,r)]\sigma(r,s)}{[2 - d(r,s)][3 - d(r,s)]},$$

where $c(r,s)$ *is given by* (3.5), $d(r,s)$ *is given by* (3.6) *and* $\sigma(r,s) = \mathrm{E}[\varepsilon_0(r)\varepsilon_0(s)]$ *for each* $r \in \mathbb{S}$ *and each* $s \in \mathbb{S}$.

Theorem 3.3. *Suppose that* $d(s) = 1$ *for each* $s \in \mathbb{S}$, $\mathrm{E}\,\varepsilon_0^2(s) < \infty$ *for each* $s \in \mathbb{S}$ *and*

$$\int_{\mathbb{S}} \sigma^2(v)\mu(\mathrm{d}v) < \infty.$$

Then

$$(\sqrt{n}\ln n)^{-1}S_n \xrightarrow{\mathcal{D}} G' \quad as \quad n \to \infty,$$

where $G' = \{G'(s) : s \in \}$ *is a zero mean Gaussian random process with the autocovariance function* $\mathrm{E}[G'(r)G'(s)] = \sigma(r,s)$, *where* $\sigma(r,s) = \mathrm{E}[\varepsilon_0(r)\varepsilon_0(s)]$ *for each* $r \in \mathbb{S}$ *and* $s \in \mathbb{S}$.

Theorem 3.4. *Suppose that* $\operatorname{ess\,inf} d > 1$ *and* $\mathrm{E}\,\varepsilon_0^2(s) < \infty$ *for each* $s \in \mathbb{S}$. *Then*

$$(\sqrt{n})^{-1}S_n \xrightarrow{\mathcal{D}} G'',$$

where $G'' = \{G'(s) : s \in \mathbb{S}\}$ *is a zero mean Gaussian random process.*

Proof of Theorem 3.2 and Theorem 3.3. The proof is based on Theorem 3.1. We begin by proving the convergence of the finite-dimensional distributions.

In order to prove the convergence of finite dimensional distributions, we investigate a sequence of random vectors

$$\begin{pmatrix} b_n^{-1}(s_1)S_n(s_1) & \dots & b_n^{-1}(s_q)S_n(s_q) \end{pmatrix}^{\mathrm{T}}, \tag{3.38}$$

where $\mathbf{x}^{\mathrm{T}}$ denotes the transpose of the vector $\mathbf{x} \in \mathbb{R}^q$, $s_1, \dots, s_q \in \mathbb{S}$ and

$$b_n(s) = \begin{cases} n^{3/2-d(s)}, & 1/2 < d(s) < 1; \\ \sqrt{n}\ln n, & d(s) = 1. \end{cases}$$

The sum of dependent random variables $S_n(s) = \sum_{k=1}^n X_k(s)$ can be expressed as a series of independent random variables: if $n \geq 2$, then we have the following identity

$$S_n(s) = \sum_{k=1}^{n} X_k(s) = \sum_{j=-\infty}^{n} z_{n,j}(s)\varepsilon_j(s),$$

where

$$z_{n,j}(s) = \begin{cases} \sum_{k=1}^{n-j+1} k^{-d(s)}, & 2 \le j \le n; \\ \sum_{k=1}^{n} (k-j+1)^{-d(s)}, & j < 2. \end{cases}$$

By denoting

$$\varepsilon_j^{(q)} = \begin{pmatrix} \varepsilon_j(s_1) & \dots & \varepsilon_j(s_q) \end{pmatrix}^{\mathrm{T}}$$

and

$$B_{n,j} = \operatorname{diag}\begin{pmatrix} b_n^{-1}(s_1) z_{n,j}(s_1) & \dots & b_n^{-1}(s_q) z_{n,j}(s_q) \end{pmatrix},$$

we can express the sequence of random vectors (3.38) compactly as

$$\sum_{j=-\infty}^{n} B_{n,j}\varepsilon_j^{(q)} = \begin{pmatrix} \sum_{j=-\infty}^{n} [b_n^{-1}(s_1) z_{n,j}(s_1)]\varepsilon_j(s_1) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \sum_{j=-\infty}^{n} [b_n^{-1}(s_q) z_{n,j}(s_q)]\varepsilon_j(s_q) \end{pmatrix}.$$

Using the fact that the operator norm of a diagonal matrix is the largest entry in absolute value, we obtain the operator norm of the matrix $B_{n,j}$

$$\|B_{n,j}\| = \max_{1 \le i \le q} |b_n^{-1}(t_i) z_{n,j}(t_i)|.$$

Now suppose that

$$\gamma_j^{(q)} := \begin{pmatrix} \gamma_j(t_1) & \dots & \gamma_j(t_q) \end{pmatrix}^{\mathrm{T}}$$

is a zero mean normal random vector with the same covariance matrix as the vector $\varepsilon_j^{(q)}$. The sequence $\sum_{j=-\infty}^{n} B_{n,j}\gamma_j^{(q)}$ converges in distribution as $n \to \infty$ to a normal random vector if and only if the sequence of covariance matrices converges. It follows from Proposition 3.5 that the sequence of covariance matrices converges.

We use Lemma 3.1 to show that the sequences

$$\Big\{ \sum_{j=-\infty}^{n} B_{n,j}\varepsilon_j^{(q)} \Big\} \quad \text{and} \quad \Big\{ \sum_{j=-\infty}^{n} B_{n,j}\gamma_j^{(q)} \Big\}$$

have the same limiting behaviour in the sense that if one converges in distribution then so does the other and their limits coincide. Propositon 3.12 establishes that the sequence of operator norms of the matrices $B_{n,j}$ satisfies two properties that are needed to apply Lemma 3.1.

Proposition 3.12. *If* $1/2 < d \leq 1$, *then both of conditions* (3.35) *and* (3.36) *are satisfied.*

Proof. To prove that condition (3.35) holds, we first notice that

$$\sup_{j\in\mathbb{Z}} \|B_{n,j}\| = \max_{1\leq i\leq q} \|b_n^{-1}(t_i) z_{n,1}(t_i)\|$$

and then we use the following asymptotic relations: if $1/2 < d < 1$, then we obtain

$$\sum_{k=1}^{n} k^{-d(t)} \sim \frac{n^{1-d(t)}}{1-d(t)};$$

if $d(s) = 1$ for each $s \in \mathbb{S}$, then we have that

$$\sum_{k=1}^{n} k^{-1} \sim \ln n.$$

We use the following expression to prove that condition (3.36) holds

$$\sum_{j=-\infty}^{n} z_{n,j}^2(t) = \sum_{j=2}^{n} \Bigg[\sum_{k=1}^{n-j+1} k^{-d(t)} \Bigg]^2 + \sum_{j=0}^{\infty} \Bigg[\sum_{k=1}^{n} (k+j)^{-d(t)} \Bigg]^2.$$

Routine approximations of sums by integrals from above lead to the following inequalities: if $1/2 < d(t) < 1$, then we have that

$$\sum_{j=2}^{n} \Bigg[\sum_{k=1}^{n-j+1} k^{-d(t)} \Bigg]^2 \leq \frac{1}{[1-d(t)]^2[3-2d(t)]} [n^{3-2d(t)} - 1];$$

if $d(t) = 1$, then we obtain

$$\sum_{j=2}^{n} \Bigg[\sum_{k=1}^{n-j+1} k^{-1} \Bigg]^2 \leq (n-1) + n \ln^2 n.$$

To prove that

$$\lim_{n\to\infty} \frac{1}{n^{3-2d(t)}} \sum_{j=0}^{\infty} \Bigg[\sum_{k=1}^{n} (k+j)^{-d(t)} \Bigg]^2 < \infty$$

for $1/2 < d(t) \leq 1$, we first divide the series in the expression above into two summands

$$\sum_{j=0}^{\infty} \Bigg[\sum_{k=1}^{n} (k+j)^{-d(t)} \Bigg]^2 = \Bigg[\sum_{k=1}^{n} k^{-d(t)} \Bigg]^2 + \sum_{j=1}^{\infty} \Bigg[\sum_{k=1}^{n} (k+j)^{-d(t)} \Bigg]^2 \tag{3.39}$$

and approximate the first summand on the right-hand side of the equation (3.39) by integral from above

$$\sum_{k=1}^{n} k^{-d(t)} \le \begin{cases} \frac{n^{1-d(t)}}{1-d(t)} & \text{if} \quad 1/2 < d(t) < 1; \\ 1 + \ln n & \text{if} \quad d(t) = 1. \end{cases}$$

We express the second summand on the right-hand side of equation (3.39) in the following way

$$\frac{1}{n^{3-2d(t)}} \sum_{j=1}^{\infty} \left[\sum_{k=1}^{n} (k+j)^{-d(t)} \right]^2 = \sum_{i=1}^{\infty} \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \left[\frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n} + \frac{j}{n} \right)^{-d(t)} \right]^2$$

and the interchange of limits leads to the result

$$\lim_{n\to\infty} \frac{1}{n^{3-2d(t)}} \sum_{j=1}^{\infty} \left[\sum_{k=1}^{n} (k+j)^{-d(t)} \right]^2 = \int_0^{\infty} \left[\int_0^1 (s+u)^{-d(t)} \mathrm{d}s \right]^2 \mathrm{d}u < \infty.$$

The proof of Proposition 3.12 is complete. □

The proof of the convergence of the finite-dimensional distributions is complete.

It follows from Remark 3.4 that

$$n^{-[3-2d(s)]} \, \mathrm{E}\, S_n^2(s) \to \mathrm{E}\, G^2(s) \quad \text{as} \quad n \to \infty$$

for each $s \in \mathbb{S}$ if $1/2 < d(s) < 1$ for each $s \in \mathbb{S}$ and

$$(\sqrt{n} \log n)^{-1} \, \mathrm{E}\, S_n^2(s) \to \mathrm{E}\, G'^2(s) \quad \text{as} \quad n \to \infty$$

for each $s \in \mathbb{S}$ if $d(s) = 1$ for each $s \in \mathbb{S}$.

Finally, we establish the existence of non-negative μ-integrable functions that dominate the sequence of the variance of the partial sums.

Proposition 3.13. *Let $s \in \mathbb{S}$. If $1/2 < d(s) < 1$, then*

$$\mathrm{E}[n^{-[3/2-d(s)]} S_n(s)]^2 \le \sigma^2(s) \left[1 + \frac{1}{2d(s)-1} \right] + \frac{\sigma^2(s) c(s)}{[1-d(s)][3-2d(s)]}, \tag{3.40}$$

where $c(s)$ is given by (3.5). *If $d(s) = 1$, then*

$$\mathrm{E}[(\sqrt{n} \ln n)^{-1} S_n(s)]^2 \le C \cdot \sigma^2(s), \tag{3.41}$$

where C is a positive constant.

Proof. To establish the first inequality in Proposition 3.13, we set $r = s$ in expression (3.17) and approximate the sums in expression (3.17) by integrals from above.

The following reasoning leads to inequality (3.41). We set $r = s$ in expression (3.17) to obtain an expression for the left-hand side of inequality (3.41). Since $d(s) = 1$ for each $s \in \mathbb{S}$, by setting $r = s$ in expression (3.4), we see that the only term in the expression of the left-hand side of inequality (3.41) that depends on s is $\sigma^2(s)$. It follows that the sequence

$$\frac{1}{\sigma^2(s)} \cdot \mathrm{E}[(\sqrt{n}\log n)^{-1} S_n(s)]^2$$

is a convergent sequence (see Remark 3.4) which does not depend on s. So it is bounded by some positive constant, say C. □

Remark 3.7. Using inequality (3.7), we obtain

$$\frac{\sigma^2(s)c(s)}{[1-d(s)][3-2d(s)]} \leq \frac{\sigma^2(s)}{[1-d(s)]^2} + \frac{\sigma^2(s)}{[1-d(s)][2d(s)-1]}.$$

If integrals (3.37) are finite, then the right-hand side of the inequality (3.40) is a μ-integrable function.

The proof of Theorem 3.2 and Theorem 3.3 is complete. □

3.3 Functional central limit theorem

We shall consider $\{\zeta_n\}$ as random elements with values in a separable Banach space $C([0,1]; L_2(\mu))$ of continuous functions $f : [0,1] \to L_2(\mu)$ endowed with the norm

$$\|f\| = \sup_{t\in[0,1]} \left[\int_{\mathbb{S}} f^2(v,t)\mu(\mathrm{d}v)\right]^{1/2}, \quad f \in C([0,1]; L_2(\mu)).$$

Before stating sufficient conditions for the functional central limit theorem, we define the limit Gaussian processes

$$\mathcal{G} = \{\mathcal{G}(s,t) : (s,t) \in \mathbb{S}\times[0,1]\} \quad \text{and} \quad \mathcal{G}' = \{\mathcal{G}'(s,t) : (s,t) \in \mathbb{S}\times[0,1]\}.$$

Let the stochastic process $\mathcal{G}$ be a restriction to $\mathbb{S}\times[0,1]$ of the stochastic process $\mathcal{X} = \{\mathcal{X}(s,t) : (s,t) \in \mathbb{S}\times[0,\infty)\}$ defined in Subsection 3.1.3. Let the stochastic process $\mathcal{G}'$ be Gaussian with the covariance function $\mathrm{E}[\mathcal{G}'(r,t)\mathcal{G}'(s,u)] = \sigma(r,s)\min(t,u)$, $(r,t),(s,u) \in \mathbb{S}\times[0,1]$. If the integral $\int_{\mathbb{S}} \sigma^2(v)\mu(\mathrm{d}v)$ is finite, then for each $t\in[0,1]$ the sample paths of the stochastic process $\{\mathcal{G}'(s,t) : s\in\mathbb{S}\}$ belong to the space $\mathcal{L}_2(\mu)$ (the proof is basically the same as the proof of Proposition 3.9).

The following proposition establishes conditions under which both of the stochastic processes $\{\mathcal{G}(\cdot,t) : t\in[0,1]\}$ and $\{\mathcal{G}'(\cdot,t) : t\in[0,1]\}$ with values in the space $L_2(\mu)$ have continuous versions.

Proposition 3.14. *If the integrals*

$$\int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)]^2}\mu(\mathrm{d}v) \quad \textit{and} \quad \int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)][2d(v)-1]}\mu(\mathrm{d}v)$$

are finite, then the $L_2(\mu)$-valued stochastic process $\{\mathcal{G}(\cdot,t) : t\in[0,1]\}$ has a continuous version.

If the integral

$$\int_{\mathbb{S}} \sigma^2(v)\mu(\mathrm{d}v)$$

is finite, then the $L_2(\mu)$-valued stochastic process $\{\mathcal{G}'(\cdot,t) : t\in[0,1]\}$ has a continuous version.

Proof. We use the following inequality for the moments of a Gaussian random element ξ with values in a separable Banach space:

$$(\mathrm{E}\,\|\xi\|^p)^{1/p} \le K_{p,q}(\mathrm{E}\,\|\xi\|^q)^{1/q}, \tag{3.42}$$

where $0<p,q<\infty$ and $K_{p,q}$ is a constant depending on p and q only (for the proof, see Ledoux and Talagrand [35], p. 59, Corollary 3.2).

Using Kolmogorov's continuity theorem (see Kallenberg [29], p. 35, Theorem 2.23), inequality (3.7) to estimate $c(s)$ from above, inequalities

$$\begin{aligned}
&\mathrm{E}\,\|\mathcal{G}(\cdot,t)-\mathcal{G}(\cdot,u)\|^4 \le \\
&\quad \le K_{4,2}^4(\mathrm{E}\,\|\mathcal{G}(\cdot,t)-\mathcal{G}(\cdot,u)\|^2)^2 \\
&\quad < K_{4,2}^4\Big[\int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)]^2}\mu(\mathrm{d}v) + \int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)][2d(v)-1]}\mu(\mathrm{d}v)\Big]^2 \cdot |t-u|^2
\end{aligned}$$

and

$$\mathrm{E}\,\|\mathcal{G}'(\cdot,t)-\mathcal{G}'(\cdot,u)\|^4 \le K_{4,2}^4\Big[\int_{\mathbb{S}}\sigma^2(v)\mu(\mathrm{d}v)\Big]^2|t-u|^2,$$

we conclude that the processes $\{\mathcal{G}(\cdot,t), t\in[0,1]\}$ and $\{\mathcal{G}'(\cdot,t), t\in[0,1]\}$ have continuous versions. □

Passing to continuous versions, we thus consider Gaussian stochastic processes $\mathcal{G}$ and $\mathcal{G}'$ as Gaussian random elements in the space $C([0,1];L_2(\mu))$. Clearly $\mathcal{G}'$ is an $L_2(\mu)$-valued Wiener process.

Now we are ready to state our main results. As usual, $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution.

Theorem 3.5. *Suppose that $1/2 < d(s) < 1$ for each $s\in\mathbb{S}$, the integrals*

$$\mathrm{E}\Big[\int_{\mathbb{S}}\frac{\varepsilon_0^2(v)}{[1-d(v)]^2}\mu(\mathrm{d}v)\Big]^{p/2} \quad \textit{and} \quad \int_{\mathbb{S}}\frac{\sigma^2(v)}{[1-d(v)][2d(v)-1]}\mu(\mathrm{d}v)$$

are finite and either $p=2$ and $\bar{d}=\operatorname{ess\,sup} d<1$ or $p>2$. Then we have that

$$n^{-H}\zeta_n \xrightarrow{\mathcal{D}} \mathcal{G} \quad \textit{as} \quad n\to\infty$$

in the space $C([0,1];L_2(\mu))$, where $\{n^{-H}\}$ is a sequence of multiplication operators given by $n^{-H}f=\{n^{-[3/2-d(s)]}f(s): s\in\mathbb{S}\}$ for $f\in L_2(\mu)$.

Remark 3.8. We have that if $1/2<d(s)<1$ for each $s\in\mathbb{S}$ and $p>0$, then

$$\mathrm{E}\,\|\varepsilon_0\|^p < 2^{-p}\,\mathrm{E}\Big[\int_{\mathbb{S}}\frac{\varepsilon_0^2(v)}{[1-d(v)]^2}\mu(\mathrm{d}v)\Big]^{p/2}$$

since $1-d(v)<1/2$.

Theorem 3.6. *Suppose that $d(s)=1$ for each $s\in\mathbb{S}$ and $\mathrm{E}\,\|\varepsilon_0\|^p<\infty$ for some $p>2$. Then we have that*

$$(\sqrt{n}\log n)^{-1}\zeta_n \xrightarrow{\mathcal{D}} \mathcal{G}' \quad \textit{as} \quad n\to\infty$$

in the space $C([0,1];L_2(\mu))$.

Theorem 3.7. *Suppose that $\underline{d}=\operatorname{ess\,inf} d>1$ and $\mathrm{E}\,\|\varepsilon_0\|^2<\infty$. Then we have that*

$$(\sqrt{n})^{-1}\zeta_n \xrightarrow{\mathcal{D}} \mathcal{G}' \quad \textit{as} \quad n\to\infty$$

in the space $C([0,1];L_2(\mu))$.

Proof of Theorem 3.7. The convergence of Theorem 3.7 follows from Theorem 5 of Račkauskas and Suquet [47] since $\sum_{j=0}^{\infty} \|u_j\| = \sum_{j=1}^{\infty} j^{-\underline{d}} < \infty$. □

Proof of Theorem 3.5 and Theorem 3.6. The proof contains two major parts. We prove the convergence of the finite-dimensional distributions of the sequences $\{n^{-H}\zeta_n\}$ and $\{(\sqrt{n}\log n)^{-1}\zeta_n\}$ in the first part and we prove the tightness of these sequences in the second part.

To avoid considerations of two separate but similar cases, we denote $b_n^{-1} = n^{-H}$ and $\zeta = \mathcal{G}$ in the proof of Theorem 3.5, whereas $b_n = \sqrt{n}\log n$ and $\zeta = \mathcal{G}'$ in the proof of Theorem 3.6.

Convergence of the finite-dimensional distributions

The convergence of the finite-dimensional distributions means that the convergence

$$\begin{pmatrix} b_n^{-1}\zeta_n(t_1) & \dots & b_n^{-1}\zeta_n(t_q) \end{pmatrix} \xrightarrow{\mathcal{D}} \begin{pmatrix} \zeta(t_1) & \dots & \zeta(t_q) \end{pmatrix} \tag{3.43}$$

holds in the space $L_2^q(\mu)$ for all $q \in \mathbb{N}$ and for all $t_1, \dots, t_q \in [0,1]$. Note that the space $L_2^q(\mu)$ is isomorphic to $L_2(\mu; \mathbb{R}^q)$, the space of $\mathbb{R}^q$-valued square μ-integrable functions with the norm

$$\|f\| = \Big[\int_{\mathbb{S}} \|f(v)\|^2 \mu(\mathrm{d}v)\Big]^{1/2}$$

for $f \in L_2(\mu; \mathbb{R}^q)$, where $||f(v)||$ denotes the Euclidean norm in $\mathbb{R}^q$.

Fix $t_1, \dots, t_q \in [0,1]$ and denote, for $s \in \mathbb{S}$,

$$\zeta_n^{(q)}(s) = (\zeta_n(t_1, s), \dots, \zeta_n(t_q, s))^{\mathrm{T}} \quad \text{and} \quad \zeta^{(q)}(s) = (\zeta(t_1, s), \dots, \zeta(t_q, s))^{\mathrm{T}},$$

where $\mathbf{x}^{\mathrm{T}}$ denotes transpose of a vector $\mathbf{x}$.

Let $\zeta_n^{(q)} = \{\zeta_n^{(q)}(s) : s \in \mathbb{S}\}$ and $\zeta^{(q)} = \{\zeta^{(q)}(s) : s \in \mathbb{S}\}$. We need to prove that

$$b_n^{-1}\zeta_n^{(q)} \xrightarrow{\mathcal{D}} \zeta^{(q)} \tag{3.44}$$

in the space $L_2(\mu; \mathbb{R}^q)$ to establish (3.43).

According to Theorem 3.1, it suffices to prove the following:

(I) there exists a measurable set $\mathbb{S}_0 \subset \mathbb{S}$ such that $\mu(\mathbb{S} \setminus \mathbb{S}_0) = 0$ and for any $p \in \mathbb{N}$ and $s_1, \ldots, s_p \in \mathbb{S}_0$ we have that

$$\left(b_n^{-1}\zeta_n^{(q)}(s_1) \quad \ldots \quad b_n^{-1}\zeta_n^{(q)}(s_p) \right) \xrightarrow{\mathcal{D}} \left(\zeta^{(q)}(s_1) \quad \ldots \quad \zeta^{(q)}(s_p) \right);$$

(II) (a) for each $s \in \mathbb{S}$,

$$\mathrm{E}\left\|b_n^{-1}\zeta_n^{(q)}(s)\right\|^2 \to \mathrm{E}\left\|\zeta^{(q)}(s)\right\|^2;$$

(b) there exists a μ-integrable function $f : \mathbb{S} \to [0, \infty)$ such that for each $s \in \mathbb{S}$ and each $n \in \mathbb{N}$

$$\mathrm{E}\left\|b_n^{-1}\zeta_n^{(q)}(s)\right\|^2 \le f(s).$$

We use Lemma 3.1 to prove (I). Let $s_1, \ldots, s_p \in \mathbb{S}$. We express the sequence

$$\left\{\left(b_n^{-1}\zeta_n^q(s_1) \quad \ldots \quad b_n^{-1}\zeta_n^q(s_p) \right)\right\}$$

of random matrices as

$$\begin{aligned}
&\left(b_n^{-1}\zeta_n^q(s_1) \quad \ldots \quad b_n^{-1}\zeta_n^q(s_p) \right) = \\
&= \sum_{j=-\infty}^{\infty} \begin{pmatrix} z_n^{-1}(s_1)a_{nj}(s_1,t_1)\varepsilon_j(s_1) & \cdots & z_n^{-1}(s_p)a_{nj}(s_p,t_1)\varepsilon_j(s_p) \\ \vdots & \ddots & \vdots \\ z_n^{-1}(s_1)a_{nj}(s_1,t_q)\varepsilon_j(s_1) & \cdots & z_n^{-1}(s_p)a_{nj}(s_p,t_q)\varepsilon_j(s_p) \end{pmatrix} \\
&= \sum_{j=-\infty}^{\infty} A_{nj}\mathcal{E}_j,
\end{aligned}$$

where

$$A_{nj} = \begin{pmatrix} z_n^{-1}(s_1)a_{nj}(s_1,t_1) & \cdots & z_n^{-1}(s_p)a_{nj}(s_p,t_1) \\ \vdots & \ddots & \vdots \\ z_n^{-1}(s_1)a_{nj}(s_1,t_q) & \cdots & z_n^{-1}(s_p)a_{nj}(s_p,t_q) \end{pmatrix},$$

$$\mathcal{E}_j = \mathrm{diag}\left(\varepsilon_j(s_1) \quad \ldots \quad \varepsilon_j(s_p) \right)$$

and

$$z_n(s) = \begin{cases} n^{3/2-d(s)}, & \text{if } 1/2 < d(s) < 1; \\ \sqrt{n}\log n, & \text{if } d(s) = 1. \end{cases}$$

If $d \in (1/2, 1]$, then the matrices $\{A_{nj}\}$ satisfy both of conditions (3.35) and (3.36). Indeed, since

$$\sup_{j\in\mathbb{Z}} a_{nj}(s,t) = a_{n1}(s,t) = \sum_{k=1}^{\lfloor nt\rfloor} k^{-d(s)} + \{nt\}(\lfloor nt\rfloor + 1)^{-d(s)},$$

we have the following asymptotic relations

$$\sup_{j\in\mathbb{Z}} a_{nj}(s,t) \sim \begin{cases} \frac{t^{1-d(s)}}{1-d(s)} \cdot n^{1-d(s)}, & \text{if } d(s) < 1; \\ \log n, & \text{if } d(s) = 1. \end{cases}$$

We have that

$$\mathrm{E}\,\zeta_n^2(s,t) = \sigma^2(s) \sum_{j=-\infty}^{\lfloor nt\rfloor+1} a_{nj}^2(s,t)$$

and we use the asymptotic behaviour of the variance $\mathrm{E}\,\zeta_n^2(s,t)$ (see Remark 3.4) to obtain the following asymptotic relations

$$\sum_{j=-\infty}^{\lfloor nt\rfloor+1} a_{nj}^2(s,t) \sim \begin{cases} \frac{c(s)}{[1-d(s)][3-2d(s)]} \cdot t^{3-2d(s)} \cdot n^{3-2d(s)}, & \text{if } 1/2 < d(s) < 1; \\ t \cdot n \log^2 n, & \text{if } d(s) = 1. \end{cases}$$

Now we investigate the sequence $\{\begin{pmatrix} b_n^{-1}\tilde{\zeta}_n^q(s_1) & \dots & b_n^{-1}\tilde{\zeta}_n^q(s_p) \end{pmatrix}\}$, which is expressed as

$$\begin{pmatrix} b_n^{-1}\tilde{\zeta}_n^q(s_1) & \dots & b_n^{-1}\tilde{\zeta}_n^q(s_p) \end{pmatrix} = \sum_{j=-\infty}^{\infty} A_{nj}\tilde{\mathcal{E}}_j, \tag{3.45}$$

where $\{\tilde{\mathcal{E}}_j\}$ is a sequence of independent and identically distributed Gaussian random matrices with zero mean and the same covariance operator as that of $\mathcal{E}_0$. Since $\{\begin{pmatrix} \tilde{\zeta}_n^q(r) & \dots & \tilde{\zeta}_n^q(s_p) \end{pmatrix}\}$ is a sequence of finite-dimensional Gaussian random elements, we only need to check for each $i = 1, \dots, p$ and each $j = 1, \dots, q$ the convergence

$$z_n^{-1}(s_i)\,\mathrm{E}\,\zeta_n(s_i,t_j) \to \mathrm{E}\,\zeta(s_i,t_j).$$

But this easily follows from Proposition 3.6 and Proposition 3.7. The proof of (I) is complete.

Next we prove (II). We prove (IIa) using equalities

$$\mathrm{E}\,\|b_n^{-1}\zeta_n^q(s)\|^2 = \sum_{i=1}^{q} \mathrm{E}[z_n^{-1}(s)\zeta_n(s,t_i)]^2 \quad \text{and} \quad \mathrm{E}\,\|\zeta^q(s)\|^2 = \sum_{i=1}^{q} \mathrm{E}\,\zeta^2(s,t_i)$$

and Remark 3.4.

An auxiliary result is needed to prove part (IIb).

Proposition 3.15. *If* $1/2 < d(s) < 1$*, then*

$$\mathrm{E}[n^{-[3/2-d(s)]}\zeta_n(s,t)]^2 \le g(s) = 2[g_1(s) + g_2(s) + g_3(s)], \tag{3.46}$$

for each $n \in \mathbb{N}$*, where*

$$g_1(s) = \sigma^2(s)\Big[1 + \frac{1}{2d(s)-1}\Big], \quad g_2(s) = \frac{\sigma^2(s)}{[1-d(s)]^2},$$

$$g_3(s) = \frac{\sigma^2(s)}{[1-d(s)][2d(s)-1]}$$

and $c(s)$ *is given by* (3.5)*.*

If $d(s) = 1$ *for each* $s \in \mathbb{S}$*, then*

$$\mathrm{E}[(\sqrt{n}\log n)^{-1}\zeta_n(s,t_i)]^2 \le M \cdot \sigma^2(s), \tag{3.47}$$

where M *is a positive constant.*

Proof. Expanding $\mathrm{E}\,\zeta_n^2(s,t)$ gives

$$\mathrm{E}\,\zeta_n^2(s,t) = \lfloor nt\rfloor\gamma_0(s) + 2\sum_{k=1}^{\lfloor nt\rfloor}(\lfloor nt\rfloor - k)\gamma_k(s) + 2\{nt\}\sum_{k=1}^{\lfloor nt\rfloor}\gamma_k(s) + \{nt\}^2\gamma_0(s). \tag{3.48}$$

Using expression (3.4) for cross-covariance, bounding series with integrals from above and using inequality (3.7) leads to the following inequalities that complete the proof of inequality (3.46):

$$\gamma_0(s) \le \sigma^2(s)\Big[1 + \frac{1}{2d(s)-1}\Big],$$

$$\sum_{k=1}^{\lfloor nt\rfloor}(\lfloor nt\rfloor - k)\gamma_k(s) \le \frac{1}{2}\Big[\frac{\sigma^2(s)}{[1-d(s)]^2} + \frac{\sigma^2(s)}{[1-d(s)][2d(s)-1]}\Big]\lfloor nt\rfloor^{3-2d(s)}$$

and

$$\sum_{k=1}^{\lfloor nt\rfloor}\gamma_k(s) \le \frac{1}{2}\Big[\frac{\sigma^2(s)}{[1-d(s)]^2} + \frac{\sigma^2(s)}{[1-d(s)][2d(s)-1]}\Big]\lfloor nt\rfloor^{2[1-d(s)]}.$$

We argue as follows to prove inequality (3.47). By setting $r = s$ in expression (3.4), we see that the only term in expression (3.48) that depends on s is $\sigma^2(s)$ since $d(s) = 1$ for each $s \in \mathbb{S}$. It follows that the sequence

$$\frac{1}{\sigma^2(s)} \cdot \mathrm{E}[(\sqrt{n}\log n)^{-1}\zeta_n(s,t)]^2$$

does not depend on s and it is a convergent sequence (see Remark 3.4). So it is bounded by some positive constant, say M. $\square$

Now we can obtain the required function f using Proposition 3.15, the fact that $\mathrm{E}\,\|\zeta^q(s)\|^2 = \sum_{i=1}^q \mathrm{E}\,\zeta^2(s,t_i)$ and setting

$$f(s) = \begin{cases} q \cdot g(s), & \text{if } d(s) < 1; \\ qM \cdot \sigma^2(s), & \text{if } d(s) = 1. \end{cases}$$

The proof of (II) is complete. This completes the proof of the convergence of the finite dimensional distributions of the sequence $\{b_n^{-1}\zeta_n\}$.

Tightness

To establish tightness of the sequence $\{b_n^{-1}\zeta_n\}$, we use Proposition 3.11. It follows from Theorem 3.2 and Theorem 3.3 that the sequence $\{b_n^{-1}S_n\}$ converges in distribution in $L_2(\mu)$. Hence the sequence $\{b_n^{-1}\zeta_n(t)\}$ also converges in distribution in $L_2(\mu)$ and the sequence $\{b_n^{-1}\zeta_n(t)\}$ is tight on $L_2(\mu)$ for every $t \in [0,1]$ and condition (i) of Proposition 3.11 holds.

Now we show that condition (ii) of Proposition 3.11 holds for the sequence $\{b_n^{-1}\zeta_n\}$. By C we denote a generic positive constant, not necessarily the same at different occurrences. We also denote

$$\Delta_n^p(t,u) = \mathrm{E}\,\|b_n^{-1}[\zeta_n(t) - \zeta_n(u)]\|^p,$$

where $p \geq 2$, $t,u \in [0,1]$ and $n \geq 1$.

Proposition 3.16. *Suppose that* $1/2 < d(s) < 1$ *for each* $s \in \mathbb{S}$ *and the integrals*

$$\int_{\mathbb{S}} \frac{\sigma^2(r)}{2d(r)-1}\mu(\mathrm{d}r) \qquad \text{and} \qquad \mathrm{E}\Big[\int_{\mathbb{S}} \frac{\varepsilon_0^2(r)}{[1-d(r)]^2}\mu(\mathrm{d}r)\Big]^{p/2}, \quad p \geq 2,$$

are finite. Let $\bar{d} = \operatorname{ess\,sup} d$. *Then*

$$\Delta_n^p(t,u) \leq C \cdot |t-u|^{(3-2\bar{d})p/2}, \quad n \geq 1. \tag{3.49}$$

Suppose that $d(s) = 1$ *for each* $s \in \mathbb{S}$ *and* $\mathrm{E}\,\|\varepsilon_0\|^p < \infty$ *for* $p \geq 2$. *Then*

$$\Delta_n^p(t,u) \leq C \cdot |t-u|^{p/2}, \quad n \geq 2. \tag{3.50}$$

We recall that $\lfloor\cdot\rfloor$ is the floor function defined by $\lfloor x\rfloor = \max\{m \in \mathbb{Z} \mid m \le x\}$ for $x \in \mathbb{R}$, $\lceil\cdot\rceil$ is the ceiling function defined by $\lceil x\rceil = \min\{m \in \mathbb{Z} \mid m \ge x\}$ for $x \in \mathbb{R}$ and $\{x\} = x - \lfloor x\rfloor$ is a fractional part of $x \in \mathbb{R}$. Observe that $\{x\} = 0$ if and only if $x \in \mathbb{Z}$ and

$$\lceil x\rceil - \lfloor x\rfloor = \begin{cases} 0, & \text{if } x \in \mathbb{Z}; \\ 1, & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}. \end{cases}$$

We need an auxiliary lemma to prove Proposition 3.16.

Lemma 3.2. *Let $0 \le u < t \le 1$, $n \ge 1$ and $\{nt\} = \{nu\} = 0$.*

If $1/2 < d(s) < 1$ for each $s \in \mathbb{S}$, then

$$n^{-[3-2d(s)]} \sum_{j=-\infty}^{nt} \Big[\sum_{k=nu+1}^{nt} v_{k-j}(s) \Big]^2 \le \Big[\frac{2}{[1-d(s)]^2} + \frac{1}{2d(s)-1} \Big] \cdot |t-u|^{3-2d(s)} \quad (3.51)$$

for $n \ge 1$, where $v_j(s)$ is given by (3.14)*.*

If $d(s) = 1$ for each $s \in \mathbb{S}$, then

$$(\sqrt{n}\log n)^{-p} \sum_{j=-\infty}^{nt} \Big[\sum_{k=nu+1}^{nt} v_{k-j} \Big]^p \le C \cdot |t-u|^{p/2}, \quad (3.52)$$

for $n \ge 2$ and $p \ge 2$, where v_j is given by (3.13)*.*

Proof. We investigate the series

$$\sum_{j=-\infty}^{nt} \Big[\sum_{k=nu+1}^{nt} v_{k-j}(s) \Big]^p \quad (3.53)$$

with $p = 2$ in the case of $1/2 < d(s) < 1$ for each $s \in \mathbb{S}$ and $p \ge 2$ in the case of $d(s) = 1$ for each $s \in \mathbb{S}$. Let us split series (3.53) into two terms

$$\sum_{j=-\infty}^{nt} \Big[\sum_{k=nu+1}^{nt} v_{k-j}(s) \Big]^p = \sum_{j=-nu+1}^{\infty} \Big[\sum_{k=nu+1}^{nt} (k+j)^{-d(s)} \Big]^p + \sum_{j=nu+1}^{nt} \Big[\sum_{k=1}^{nt-j+1} k^{-d(s)} \Big]^p \quad (3.54)$$

and then split the first term on the right-hand side of (3.54) again into two terms

$$\sum_{j=-nu+1}^{\infty} \Big[\sum_{k=nu+1}^{nt} (k+j)^{-d(s)} \Big]^p = \sum_{j=-nu+1}^{n(t-2u)} \Big[\sum_{k=nu+1}^{nt} (k+j)^{-d(s)} \Big]^p$$

$$+\sum_{j=n(t-2u)+1}^{\infty}\Big[\sum_{k=nu+1}^{nt}(k+j)^{-d(s)}\Big]^p. \quad (3.55)$$

The first term on the right-hand side of (3.55) is estimated from above in the following way:

$$n^{-[3-2d(s)]}\sum_{j=-nu+1}^{n(t-2u)}\Big[\sum_{k=nu+1}^{nt}(k+j)^{-d(s)}\Big]^2 \le \frac{n|t-u|}{n^{3-2d(s)}}\Big[\sum_{k=nu+1}^{nt}(k-nu)^{-d(s)}\Big]^2$$

$$\le \frac{|t-u|^{3-2d(s)}}{[1-d(s)]^2}$$

if $1/2 < d(s) < 1$ for each $s \in \mathbb{S}$;

$$(\sqrt{n}\log n)^{-p}\sum_{j=-nu+1}^{n(t-2u)}\Big[\sum_{k=nu+1}^{nt}(k+j)^{-1}\Big]^p \le \Big[\frac{1}{\log n}\sum_{k=nu+1}^{nt}(k-nu)^{-1}\Big]^p\cdot|t-u|^{p/2}$$

$$\le \Big[\frac{1+\log(n|t-u|)}{\log n}\Big]^p\cdot|t-u|^{p/2}$$

if $d(s)=1$ for each $s\in\mathbb{S}$ since $1/n \le |t-u|$ (otherwise $t=u$ because we assume that $\{nt\}=\{nu\}=0$). The second term on the right-hand side of (3.55) is estimated from above using the inequality

$$\sum_{j=n(t-2u)+1}^{\infty}\Big[\sum_{k=nu+1}^{nt}(k+j)^{-d(s)}\Big]^p \le (n|t-u|)^p\sum_{j=n(t-2u)+1}^{\infty}(nu+j)^{-pd(s)}$$

$$\le \frac{(n|t-u|)^{p+1-pd(s)}}{pd(s)-1}$$

and observing that

$$n^{-[3-2d(s)]}(n|t-u|)^{p+1-pd(s)} = |t-u|^{3-2d(s)}$$

if $1/2 < d(s) < 1$ for each $s\in\mathbb{S}$ and $p=2$ and

$$(\sqrt{n}\log n)^{-p}(n|t-u|)^{p+1-pd(s)} \le \frac{|t-u|^{p/2}}{\log^p 2}$$

if $d(s)=1$ for each $s\in\mathbb{S}$ and $p\ge 2$ since $1/n\le|t-u|$.

The second term on the right-hand side of (3.54) is estimated in the following way:

$$n^{-[3-2d(s)]}\sum_{j=nu+1}^{nt}\Big[\sum_{k=1}^{nt-j+1}k^{-d(s)}\Big]^2 \le \frac{1}{[1-d(s)]^2[3-2d(s)]}\cdot|t-u|^{3-2d(s)}$$

if $1/2 < d(s) < 1$ for each $s \in \mathbb{S}$ and

$$(\sqrt{n}\log n)^{-p}\sum_{j=nu+1}^{nt}\Big[\sum_{k=1}^{nt-j+1}k^{-1}\Big]^p \le \frac{1}{n^{p/2}\log^p n}\sum_{j=nu+1}^{nt}[1+\log(nt-j+1)]^p$$
$$\le 2\Big[\frac{1}{\log 2}+\frac{\log(n|t-u|)}{\log n}\Big]^p\cdot|t-u|^{p/2}$$

if $d(s) = 1$ for each $s \in \mathbb{S}$. The proof of Lemma 3.2 is complete. □

Now we are ready to prove Proposition 3.16.

Proof of Proposition 3.16. Let $t, u \in [0, 1]$. There is no loss of generality by assuming that $t > u$. Set $t' = \lfloor nt\rfloor/n$ and $u' = \lceil nu\rceil/n$, so that $t, t' \in [\lfloor nt\rfloor/n, \lceil nt\rceil/n]$, $u, u' \in [\lfloor nu\rfloor/n, \lceil nu\rceil/n]$, $\{nt'\} = \{nu'\} = 0$ and $|t' - u'| \le |t - u|$. Since

$$\Delta_n^p(t,u) \le C[\Delta_n^p(t,t') + \Delta_n^p(t',u') + \Delta_n^p(u',u)],$$

we can establish inequalities (3.49) and (3.50) by investigating two cases: either $t, u \in [\kappa/n, (\kappa + 1)/n]$ for some $\kappa \in \{0, \dots, n - 1\}$ or $\{nt\} = \{nu\} = 0$.

First, suppose that $t, u \in [\kappa/n, (\kappa + 1)/n]$ for some $\kappa \in \{0, \dots, n - 1\}$. Then $|t - u| \le 1/n$ and

$$\zeta_n(t) - \zeta_n(u) = n|t-u|X_{\kappa+1},$$

so that

$$\Delta_n^p(t,u) \le [n|t-u|]^p\|n^{-H}\|^p\,\mathrm{E}\,\|X_0\|^p \le \mathrm{E}\,\|X_0\|^p\cdot|t-u|^{(3-2\bar{d})p/2}$$

if $1/2 < d(s) < 1$ for each $s \in \mathbb{S}$ and

$$\Delta_n^p(t,u) = [n|t-u|]^p(\sqrt{n}\log n)^{-p}\,\mathrm{E}\,\|X_0\|^p \le \frac{\mathrm{E}\,\|X_0\|^p}{\log^p 2}\cdot|t-u|^{p/2}$$

if $d(s) = 1$ for each $s \in \mathbb{S}$ and $n \ge 2$.

We have that

$$\mathrm{E}\,\|X_0\|^p \le 2^{p-1}\Big(C\frac{p}{\log p}\Big)^p\Big[(\mathrm{E}\,\|X_0\|^2)^{p/2} + \sum_{j=0}^{\infty}\mathrm{E}\,\|u_j\varepsilon_{k-j}\|^p\Big] \tag{3.56}$$

by using a slight modification of the inequality stated in Theorem 6.20 of Ledoux and Talagrand [35]. Since

$$\mathrm{E}\,\|X_0\|^2 \le \mathrm{E}\,\|\varepsilon_0\|^2 + \int_{\mathbb{S}}\frac{\sigma^2(r)}{2d(r)-1}\mu(\mathrm{d}r)$$

and $\sum_{j=0}^{\infty} \mathrm{E}\,\|u_j\varepsilon_{k-j}\|^p \le \mathrm{E}\,\|\varepsilon_0\|^p \sum_{j=1}^{\infty} j^{-p/2}$, we have that $\mathrm{E}\,\|X_0\|^p < \infty$.

Secondly, suppose that $\{nt\} = \{nu\} = 0$. Then $|t-u| \ge 1/n$ ($\Delta_n^p(t,u) = 0$ if $\{nt\} = \{nu\} = 0$ and $|t-u| < 1/n$). The increment $b_n^{-1}[\zeta_n(t) - \zeta_n(u)]$ may be expressed as a series of independent $L_2(\mu)$-valued random elements

$$b_n^{-1}[\zeta_n(t) - \zeta_n(u)] = \sum_{j=-\infty}^{nt} b_n^{-1} \sum_{k=nu+1}^{nt} v_{k-j}\varepsilon_j$$

where v_j is given by (3.14). Using the same inequality as in (3.56), we have that

$$\Delta_n^p(t,u) \le 2^{p-1}\Big(C\frac{p}{\log p}\Big)^p \Big[\Delta_n^p 2^{p/2}(t,u) + \sum_{j=-\infty}^{nt} \mathrm{E}\Big\|b_n^{-1} \sum_{k=nu+1}^{nt} v_{k-j}\varepsilon_j\Big\|^p\Big].$$

If $1/2 < d(s) < 1$ for each $s \in \mathbb{S}$, then we have that

$$\Delta_n^2(t,u) = \int_{\mathbb{S}} \sigma^2(r) n^{-[3-2d(r)]} \sum_{j=-\infty}^{nt} \Big[\sum_{k=nu+1}^{nt} v_{k-j}(r)\Big]^2 \mu(\mathrm{d}r) \tag{3.57}$$

and

$$\sum_{j=-\infty}^{nt} \mathrm{E}\Big\|b_n^{-1} \sum_{k=nu+1}^{nt} v_{k-j}\varepsilon_j\Big\|^p =$$

$$= \sum_{j=-\infty}^{nt} \mathrm{E}\Big[\int_{\mathbb{S}} n^{-[3-2d(r)]}\Big|\sum_{k=nu+1}^{nt} v_{k-j}(r)\Big|^2 \varepsilon_j^2(r)\mu(\mathrm{d}r)\Big]^{p/2}. \tag{3.58}$$

If $d(s) = 1$ for each $s \in \mathbb{S}$, then we obtain

$$\Delta_n^2(t,u) = \mathrm{E}\,\|\varepsilon_0\|^2(\sqrt{n}\log n)^{-2} \sum_{j=-\infty}^{nt} \Big[\sum_{k=nu+1}^{nt} v_{k-j}\Big]^2 \tag{3.59}$$

and

$$\sum_{j=-\infty}^{nt} \mathrm{E}\Big\|b_n^{-1} \sum_{k=nu+1}^{nt} v_{k-j}\varepsilon_j\Big\|^p t = \mathrm{E}\,\|\varepsilon_0\|^p(\sqrt{n}\log n)^{-p} \sum_{j=-\infty}^{nt} \Big[\sum_{k=nu+1}^{nt} v_{k-j}\Big]^p. \tag{3.60}$$

We estimate (3.57), (3.59) and (3.60) using Lemma 3.2 and we need to estimate series (3.58) for $p > 2$ when $1/2 < d(s) < 1$ for each $s \in \mathbb{S}$. As in (3.54) and (3.55), we split series (3.58) into three parts and estimate them from above separately. The estimation is essentially similar to the estimation of series (3.53). Let us recall that we assume that $1/n \le |t-u|$ if $\{nt\} = \{nu\} = 0$. The following inequalities are obtained:

$$\sum_{j=-nu+1}^{n(t-2u)} \mathrm{E}\Big[\int_{\mathbb{S}} \frac{|\sum_{k=nu+1}^{nt}(k+j)^{-d(r)}|^2 \varepsilon_j^2(r)}{n^{3-2d(r)}}\mu(\mathrm{d}r)\Big]^{p/2} \le$$
$$\le \mathrm{E}\Big[\int_{\mathbb{S}} \frac{\varepsilon_0^2(r)}{[1-d(r)]^2}\mu(\mathrm{d}r)\Big]^{p/2}|t-u|^{(3-2\bar{d})p/2}$$

since

$$\frac{\sum_{k=nu+1}^{nt}(k-nu)^{-d(r)}}{n^{1-d(r)}} \le \frac{|t-u|^{1-d(r)}}{1-d(r)};$$

$$\sum_{j=n(t-2u)+1}^{\infty} \mathrm{E}\Big[\int_{\mathbb{S}} \frac{|\sum_{k=nu+1}^{nt}(k+j)^{-d(r)}|^2 \varepsilon_j^2(r)}{n^{3-2d(r)}}\mu(\mathrm{d}r)\Big]^{p/2} \le \frac{\mathrm{E}\,\|\varepsilon_0\|^p}{p/2-1}|t-u|^{(3-2\bar{d})p/2}$$

since

$$\Big(\frac{n}{nu+j}\Big)^{2d(r)} = \Big(\frac{n}{n|t-u|}\Big)^{2d(r)}\Big(\frac{n|t-u|}{nu+j}\Big)^{2d(r)} \le n|t-u|^{1-2\bar{d}}(nu+j)^{-1}$$

for $j \ge n(t-2u)+1$;

$$\sum_{j=nu+1}^{nt} \mathrm{E}\Big[\int_{\mathbb{S}} \frac{|\sum_{k=1}^{nt-j+1} k^{-d(r)}|^2 \varepsilon_j^2(r)}{n^{3-2d(r)}}\mu(\mathrm{d}r)\Big]^{p/2}$$
$$\le \frac{2^{1+p(1-\bar{d})}}{1+p(1-\bar{d})}\,\mathrm{E}\Big[\int_{\mathbb{S}} \frac{\varepsilon_0^2(r)}{[1-d(r)]^2}\mu(\mathrm{d}r)\Big]^{p/2}|t-u|^{(3-\bar{d})p/2} \quad (3.61)$$

since

$$\frac{\sum_{k=1}^{nt-j+1} k^{-d(r)}}{n^{1-d(r)}} \le \frac{1}{1-d(r)}\Big[\frac{nt-j+1}{n}\Big]^{1-d(r)} \le \frac{1}{1-d(r)}\Big[\frac{nt-j+1}{n}\Big]^{1-\bar{d}}$$

for $nu+1 \le j \le nt$.

The proof of Proposition 3.16 is complete. □

We established the convergence of the finite-dimensional distributions and the tightness of the sequence $\{b_n^{-1}\zeta_n\}$. The proof of Theorem 3.5 and Theorem 3.6 is complete. □

4 Law of large numbers

Suppose that $\mathbb{H}$ is a separable Hilbert space. We investigate the law of large numbers for an $\mathbb{H}$-valued linear process $\{X_k\} = \{X_k : k \in \mathbb{Z}\}$ defined by

$$X_k = \sum_{j=0}^{\infty} a_j(\varepsilon_{k-j}) \tag{4.1}$$

for each $k \in \mathbb{Z}$, where $\{a_j\} = \{a_j : j \geq 0\}$ are bounded linear operators from $\mathbb{H}$ to $\mathbb{H}$ and $\{\varepsilon_k\} = \{\varepsilon_k : k \in \mathbb{Z}\}$ are independent and identically distributed $\mathbb{H}$-valued random elements.

Let us define the normalizing sequence $\{b_n(p)\} = \{b_n(p) : n \geq 1\}$ for $p \geq 1$ before we state our results. For $j \in \mathbb{Z}$, denote

$$\tilde{a}_j = \begin{cases} a_j & \text{if } j \geq 0, \\ 0 & \text{if } j < 0. \end{cases}$$

Then we have that

$$S_n = \sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \sum_{j=-\infty}^{k} a_{k-j}\varepsilon_j = \sum_{k=1}^{n} \sum_{j=-\infty}^{n} \tilde{a}_{k-j}\varepsilon_j = \sum_{j=-\infty}^{n} w_{nj}\varepsilon_j$$

for $n \geq 1$, where

$$w_{nj} = \sum_{k=1}^{n} \tilde{a}_{k-j} \tag{4.2}$$

for $n \geq 1$ and $j \in \mathbb{Z}$. Let us observe that sum (4.2) contains at most $\min\{n - j + 1, n\}$ non-zero terms for $j \leq n$.

Denote

$$b_n(p) = \left(\sum_{j=-\infty}^{n} \|w_{nj}\|^p \right)^{1/p} \tag{4.3}$$

for $n \geq 1$ and $p > 0$. We have that

$$b_n(p) = \left(\sum_{j=-\infty}^{n} \|w_{nj}\|^p \right)^{1/p} = \lim_{N\to\infty} \left(\sum_{j=-N}^{n} \left\| \sum_{k=1}^{n} \tilde{a}_{k-j} \right\|^p \right)^{1/p}$$

and

$$\lim_{N\to\infty}\Big(\sum_{j=-N}^{n}\Big\|\sum_{k=1}^{n}\tilde{a}_{k-j}\Big\|^p\Big)^{1/p}\le\lim_{N\to\infty}\sum_{k=1}^{n}\Big(\sum_{j=-N}^{n}\|\tilde{a}_{k-j}\|^p\Big)^{1/p}=n\Big(\sum_{j=0}^{\infty}\|a_j\|^p\Big)^{1/p}$$

using the triangle inequality for the norm

$$\|\mathbf{K}\|=\Big(\sum_{i=1}^{d}\|K_i\|^p\Big)^{1/p}$$

for $p\ge 1$ and $\mathbf{K}\in L^d(\mathbb{H})$. Hence, $b_n(p)$ is finite for each $n\ge 1$ and each $p\ge 1$ if $\sum_{j=0}^{\infty}\|a_j\|^p<\infty$.

Now we are ready to state our main results. We begin with the case when $\sum_{j=0}^{\infty}\|a_j\|$ is finite.

Theorem 4.1. *Let* $1<p<2$. *Suppose that* $\sum_{j=0}^{\infty}\|a_j\|<\infty$, $x^p\Pr\{\|\varepsilon_0\|>x\}\to 0$ *as* $x\to\infty$ *and* $\mathrm{E}\,\varepsilon_0=0$. *Then*

$$\frac{S_n}{n^{1/p}}\to 0$$

in probability as $n\to\infty$.

Theorem 4.2. *Let* $1<p<2$. *Suppose that* $\sum_{j=0}^{\infty}\|a_j\|<\infty$, $\mathrm{E}\,\|\varepsilon_0\|^p<\infty$ *and* $\mathrm{E}\,\varepsilon_0=0$. *Then*

$$\frac{S_n}{n^{1/p}}\to 0$$

almost surely as $n\to\infty$.

Let us turn to the case when $\sum_{j=0}^{\infty}\|a_j\|$ is not necessarily finite.

Theorem 4.3. *Let* $1<p<2$. *Suppose that* $\sum_{j=0}^{\infty}\|a_j\|^p<\infty$, $x^p\Pr\{\|\varepsilon_0\|>x\}\to 0$ *as* $x\to\infty$ *and* $\mathrm{E}\,\varepsilon_0=0$. *If*

$$\lim_{n\to\infty}\frac{\sup_{j\le n}\|w_{nj}\|}{b_n(p)}=0,$$

then

$$\frac{S_n}{b_n(p)}\to 0$$

in probability as $n\to\infty$.

Since $\sup_{j\le n}\|w_{nj}\|\le b_n(q)$ for $q\ge 1$, we have the following corollary of Theorem 4.3.

Corollary 4.1. *Let $1 < p < 2$. Suppose that $\sum_{j=0}^{\infty} \|a_j\|^p < \infty$, $x^p \Pr\{\|\varepsilon_0\| > x\} \to 0$ as $x \to \infty$ and $\mathrm{E}\,\varepsilon_0 = 0$. If*

$$\lim_{n\to\infty} \frac{b_n(q)}{b_n(p)} = 0$$

for some $q > p$, then

$$\frac{S_n}{b_n(p)} \to 0$$

in probability as $n \to \infty$.

Theorem 4.4. *Let $1 \le p < 2$. Suppose that $\sum_{j=0}^{\infty} \|a_j\|^p < \infty$, $\mathrm{E}[\|\varepsilon_0\|^p \log(1 + \|\varepsilon_0\|)] < \infty$ and $\mathrm{E}\,\varepsilon_0 = 0$. If*

$$\frac{b_n(q)}{b_n(p)} = O(n^{1/q-1/p}) \tag{4.4}$$

as $n \to \infty$ for some $p < q \le 2$, then

$$\frac{S_n}{b_n(p)} \to 0$$

almost surely as $n \to \infty$.

4.1 Examples

We give two examples of particular operators $\{a_j\}$ and establish asymptotic behaviour of the normalizing sequences $\{b_n(p)\}$ when the series $\sum_{j=0}^{\infty} \|a_j\|$ diverges.

Let us observe that the normalizing sequence $\{b_n(p)\}$ can be expressed explicitly in terms of the operators $\{a_j\}$

$$\begin{aligned} b_n^p(p) &= \sum_{j=-\infty}^{0} \left\| \sum_{k=1}^{n} \tilde{a}_{k-j} \right\|^p + \sum_{j=1}^{n} \left\| \sum_{k=1}^{n} \tilde{a}_{k-j} \right\|^p \\ &= \sum_{j=-\infty}^{0} \left\| \sum_{k=1}^{n} a_{k-j} \right\|^p + \sum_{j=1}^{n} \left\| \sum_{k=0}^{n-j} a_k \right\|^p . \end{aligned} \tag{4.5}$$

Proposition 4.1. *Let $p > 1$ and $1/p < d < 1$. Suppose that $\mathbb{H} = \mathbb{R}$ and $a_j = (j+1)^{-d}$ for each $j \ge 0$. Then*

$$b_n(p) \sim c \cdot n^{1/p+1-d}$$

as $n \to \infty$, where c is a positive constant.

Let us compare our results with the results of Louhichi and Soulier [37] (see Theorem 2.14 in Chapter 2). Suppose that $1 \le s < \alpha < 2$, $a_j = (j+1)^{-d}$ for each $j \ge 0$ with some $1/s < d < 1$, $\mathrm{E}\,\varepsilon_0 = 0$ and $\mathrm{E}[|\varepsilon_0|^p \log(1+|\varepsilon_0|)] < \infty$ for all $s \le p < \alpha$. Then it follows from our results (Theorem 4.3 and Proposition 4.1) that

$$\frac{S_n}{n^{1/p+1-1/s}} \to 0$$

almost surely as $n \to \infty$ for all $p \le \alpha$ since $1/s < d$. The advantage of our result is that we do not need to assume that $\{\varepsilon_k\}$ are symmetric α-stable random variables. The advantage of the results of Louhichi and Soulier [37] is that they do not need to assume that $a_j = (j+1)^{-d}$ for each $j \ge 0$, they only need to assume that $\sum_{j=0}^{\infty} |a_j|^s < \infty$ for some $1 \le s < \alpha$.

Proof of Proposition 4.1. We have that

$$\begin{aligned} b_n^p(p) &= \sum_{j=-\infty}^{0} \left| \sum_{k=1}^{n} (k-j+1)^{-d} \right|^p + \sum_{j=1}^{n} \left| \sum_{k=0}^{n-j} (k+1)^{-d} \right|^p \\ &= \sum_{j=1}^{\infty} \left| \sum_{k=1}^{n} (k+j)^{-d} \right|^p + \sum_{j=1}^{n} \left| \sum_{k=1}^{n-j+1} k^{-d} \right|^p \end{aligned}$$

using expression (4.5).

We obtain that the limit

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n^{1+p(1-d)}} \sum_{j=1}^{\infty} \left| \sum_{k=1}^{n} (k+j)^{-d} \right|^p &= \lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{1}{n} \sum_{l=(j-1)n+1}^{jn} \left| \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n} + \frac{l}{n} \right)^{-d} \right|^p \\ &= \sum_{j=1}^{\infty} \lim_{n\to\infty} \frac{1}{n} \sum_{l=(j-1)n+1}^{jn} \left| \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n} + \frac{l}{n} \right)^{-d} \right|^p \\ &= \sum_{j=1}^{\infty} \int_{j-1}^{j} \left| \int_0^1 (x+y)^{-d} \mathrm{d}x \right|^p \mathrm{d}y \\ &= \int_0^{\infty} \left| \int_0^1 (x+y)^{-d} \mathrm{d}x \right|^p \mathrm{d}y \qquad (4.6) \end{aligned}$$

is finite since both of the integrals

$$\int_0^1 \left| \int_0^1 (x+y)^{-d} \mathrm{d}x \right|^p \mathrm{d}y \le \left| \int_0^1 x^{-d} \mathrm{d}s \right|^p$$

and

$$\int_1^{\infty} \left| \int_0^1 (x+y)^{-d} \mathrm{d}x \right|^p \mathrm{d}y \le \int_1^{\infty} y^{-pd} \mathrm{d}y$$

are finite. Also, the limit is positive since integral (4.6) is positive.

By approximating sums with definite integrals, we have that

$$\sum_{k=1}^{n-j+1} k^{-d} \leq \frac{(n-j+1)^{1-d}}{1-d}$$

and

$$\sum_{j=1}^{n} \left| \frac{(n-j+1)^{1-d}}{1-d} \right|^p \leq \int_0^n \frac{(n-x+1)^{p(1-d)}}{[1-d]^p} \mathrm{d}x = \frac{(n+1)^{1+p(1-d)}-1}{[1+p(1-d)][1-d]^p}.$$

Hence,

$$\lim_{n\to\infty} \frac{1}{n^{1+p(1-d)}} \sum_{j=1}^{n} \left| \sum_{k=1}^{n-j+1} k^{-d} \right|^p < \infty.$$

The proof is complete. □

Now we establish asymptotic behaviour of the normalizing sequence $\{b_n(p)\}$ when the operators $\{a_j\}$ are given by (3.2).

Proposition 4.2. *Let $\mathbb{H} = L_2(\mu)$ and $\{a_j\}$ defined by*

$$a_j = (j+1)^{-D}$$

for each $j \geq 0$, where D is a multiplication operator such that $Df = \{d(s)f(s) : s \in \mathbb{S}\}$ for each $f \in L_2(\mu)$ and $d : \mathbb{S} \to \mathbb{R}$ is a measurable function. Denote $\underline{d} = \operatorname{ess\,inf}_{s\in\mathbb{S}} d(s)$. If $1/p < \underline{d} < 1$, then

$$b_n(p) \sim c \cdot n^{1/p+1-\underline{d}}$$

as $n \to \infty$, where c is a positive constant.

Proof. By equation (4.5), we have that

$$\begin{aligned} b_n^p(p) &= \sum_{j=-\infty}^{0} \left\| \sum_{k=1}^{n} (k-j+1)^{-D} \right\|^p + \sum_{j=1}^{n} \left\| \sum_{k=0}^{n-j} (k+1)^{-D} \right\|^p \\ &= \sum_{j=1}^{\infty} \left\| \sum_{k=1}^{n} (k+j)^{-D} \right\|^p + \sum_{j=1}^{n} \left\| \sum_{k=1}^{n-j+1} k^{-D} \right\|^p. \end{aligned}$$

Since

$$\left\| \sum_{k=1}^{n} (k+j)^{-D} \right\| = \operatorname*{ess\,sup}_{s\in\mathbb{S}} \left[\sum_{k=1}^{n} (k+j)^{-d(s)} \right] = \sum_{k=1}^{n} (k+j)^{-\underline{d}}$$

and

$$\left\| \sum_{k=1}^{n-j+1} k^{-D} \right\| = \operatorname*{ess\,sup}_{s \in \mathbb{S}} \left[\sum_{k=1}^{n-j+1} k^{-d(s)} \right] = \sum_{k=1}^{n-j+1} k^{-\underline{d}}$$

(see Conway [9], p. 28), we have that

$$b_n^p(p) = \sum_{j=1}^{\infty} \left| \sum_{k=1}^{n} (k+j)^{-\underline{d}} \right|^p + \sum_{j=1}^{n} \left| \sum_{k=1}^{n-j+1} k^{-\underline{d}} \right|^p.$$

Hence,

$$b_n(p) \sim c \cdot n^{1/p+1-\underline{d}}$$

using Proposition 4.1, where c is a positive constant. The proof is complete. □

It seems that $b_n(p) = O(n^{1/p})$ as $n \to \infty$ if the series $\sum_{j=0}^{\infty} \|a_j\|$ converges, but we cannot prove this statement for all $p \geq 1$. However, we can prove this statement for $p = 2$ and $\mathbb{H} = \mathbb{R}$.

Lemma 4.1. *Suppose that* $\sum_{j=0}^{\infty} |a_j| < \infty$. *Then* $b_n(2) = O(n^{1/2})$ *as* $n \to \infty$, *where* $b_n(2)$ *is given by* (4.3).

Proof. Suppose that $\mathrm{E}\,\varepsilon_0 = 0$ and $0 < \mathrm{E}\,\varepsilon_0^2 < \infty$. Using the stationarity of $\{X_k\}$,

$$\begin{aligned} \mathrm{E}\left| \sum_{k=1}^{n} X_k \right|^2 &= n \left[-\mathrm{E}\,X_0^2 + 2 \sum_{k=0}^{n-1} (1 - k/n)\, \mathrm{E}[X_0 X_k] \right] \\ &= n\, \mathrm{E}\,\varepsilon_0^2 \left[-\sum_{j=0}^{\infty} a_j^2 + 2 \sum_{k=0}^{n-1} (1 - k/n) \sum_{j=0}^{\infty} a_j a_{j+k} \right]. \end{aligned}$$

Using the fact that

$$\mathrm{E}\left| \sum_{k=1}^{n} X_k \right|^2 = \mathrm{E}\left| \sum_{j=-\infty}^{n} w_{nj} \varepsilon_j \right|^2 = \mathrm{E}\,\varepsilon_0^2 \sum_{j=-\infty}^{n} w_{nj}^2,$$

we obtain

$$b_n^2(2) = \sum_{j=-\infty}^{n} w_{nj}^2 = n \left[-\sum_{j=0}^{\infty} a_j^2 + 2 \sum_{k=0}^{n-1} (1 - k/n) \sum_{j=0}^{\infty} a_j a_{j+k} \right].$$

If $\sum_{j=0}^{\infty} |a_j| < \infty$, then $\sum_{k=0}^{\infty} \left| \sum_{j=0}^{\infty} a_j a_{j+k} \right| < \infty$ (see Hamilton [23], p. 70). Also,

$$\lim_{n \to \infty} \sum_{k=0}^{n-1} (1 - k/n) \sum_{j=0}^{\infty} a_j a_{j+k} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_j a_{j+k}$$

since any convergent series is Cesàro summable, and the sum of the series agrees with its Cesàro sum. Hence, $b_n(2) = O(n^{1/2})$ as $n \to \infty$. □

4.2 Preliminaries

4.2.1 Moments of random elements

For $0 < p < \infty$, $L_p = L_p(\Omega, \mathcal{F}, \Pr; \mathbb{H})$ denotes the space of all $\mathbb{H}$-valued random elements ξ on $(\Omega, \mathcal{F}, \Pr)$ such that

$$\|\xi\|_p = (\mathrm{E}\,\|\xi\|^p)^{1/p} < \infty$$

and $L_{p,\infty} = L_{p,\infty}(\Omega, \mathcal{F}, \Pr; \mathbb{H})$ denotes the space of all $\mathbb{H}$-valued random variables ξ on $(\Omega, \mathcal{F}, \Pr)$ such that

$$\|\xi\|_{p,\infty} = \big(\sup_{x>0}(x^p \Pr\{\|\xi\| > x\})\big)^{1/p} < \infty.$$

The functional $\|\cdot\|_{p,\infty}$ is only a quasi-norm, i.e. it satisfies the norm axioms, except that the triangle inequality is replaced by

$$\|\zeta + \xi\|_{p,\infty} \le \max\{2, 2^{1/p}\}(\|\zeta\|_{p,\infty} + \|\xi\|_{p,\infty}) \tag{4.7}$$

for $\zeta, \xi \in L_{p,\infty}$.

We have, when $r > p > 0$,

$$\|\xi\|_{p,\infty} \le \|\xi\|_p \le \Big(\frac{r}{r-p}\Big)^{1/p} \|\xi\|_{r,\infty}, \tag{4.8}$$

hence $L_{r,\infty} \subset L_p \subset L_{p,\infty}$ (actually, if ξ is in L_p, then $\lim_{x\to\infty} x^p \Pr\{\|\xi\| > x\} = 0$). There exists a constant $C > 0$ with $1 \le p < 2$ such that

$$\Big\|\sum_{i=1}^{n} \xi_i\Big\|_{p,\infty}^{p} \le C \sum_{i=1}^{n} \|\xi_i\|_{p,\infty}^{p} \tag{4.9}$$

for independent and symmetric $\mathbb{H}$-valued random elements $\xi_1, \ldots, \xi_n \in L_{p,\infty}$ (see Section 5.6 in Lin and Bai [36] and Proposition 9.13 of Ledoux and Talagrand [35]).

We need an auxiliary lemma.

Lemma 4.2. *Let ξ be a non-negative random variable. For any $q > 0$ and $c > 0$,*

$$\mathrm{E}[\xi^q I_{\{\xi \le c\}}] = q \int_0^c s^{q-1} \Pr\{\xi > s\} \mathrm{d}s - c^q \Pr\{\xi > c\}.$$

Similar result to Lemma 4.2 is used in the proof of Feller's weak law of large numbers without a first moment assumption (see Section VII.7 of Feller [17] and Theorem 7.2.1 of Resnick [51] for details).

Proof of Lemma 4.2. We have that

$$\begin{aligned} \mathrm{E}[\xi^q I_{\{\xi \le c\}}] &= \int_{\{x:x\le c\}} x^q \mathrm{d}F(x) = \int_{x\le c} \left[\int_0^x q s^{q-1} \mathrm{d}s\right] \mathrm{d}F(x) \\ &= q\int_0^c s^{q-1} \left[\int_{s<x\le a} \mathrm{d}F(x)\right] \mathrm{d}s \\ &= q\int_0^c s^{q-1} \Pr\{\xi > s\} \mathrm{d}s - c^q \Pr\{\xi > c\} \end{aligned}$$

by applying Fubini's theorem. □

4.2.2 Convergence of the series of random elements

We investigate two separate cases: real-valued case and $\mathbb{H}$-valued case. The real-valued case might be of independent interest since we establish almost sure convergence of series (4.1) under the assumption of finite $\|\varepsilon_0\|_{p,\infty}$ for all $p > 0$. Of particular interest is the case when $p = 1$ and the case when $p = 2$. The proof of the real-valued case uses the fact that $|a_j \varepsilon_0| = |a_j||\varepsilon_0|$ for each $j \ge 0$, which is not true for general Hilbert spaces, i.e. $\|a_j(\varepsilon_0)\| \ne \|a_j\|\|\varepsilon_0\|$, we only have that $\|a_j(\varepsilon_0)\| \le \|a_j\|\|\varepsilon_0\|$.

Real-valued case

Proposition 4.3. *Let $p > 0$. Suppose that $\|\varepsilon_0\|_{p,\infty} < \infty$ and $\mathrm{E}\,\varepsilon_0 = 0$ if $p > 1$. Series* (4.1) *converges almost surely if:*

(a) $p \ne 1$, $p \ne 2$ and $\sum_{j=0}^{\infty} |a_j|^q < \infty$, where $q = \min\{p, 2\}$;

(b) $p = 1$ and $\sum_{j=0}^{\infty} |a_j| \log |a_j|^{-1} < \infty$;

(c) $p = 2$ and $\sum_{j=0}^{\infty} |a_j|^2 \log |a_j|^{-1} < \infty$.

Proof. We use Kolmogorov's three-series theorem to establish the almost sure convergence. Assume without loss of generality that $a_j \ne 0$ for each $j \ge 0$. We establish the convergence of the following series:

$$\sum_{j=0}^{\infty} \Pr\{|\varepsilon_0| > |a_j|^{-1}\}; \tag{4.10}$$

$$\sum_{j=0}^{\infty} a_j \operatorname{E}[\varepsilon_0 I_{\{|\varepsilon_0| \le |a_j|^{-1}\}}]; \tag{4.11}$$

$$\sum_{j=0}^{\infty} a_j^2 \operatorname{Var}[\varepsilon_0 I_{\{|\varepsilon_0| \le |a_j|^{-1}\}}]. \tag{4.12}$$

First, we establish convergence of series (4.10). We have that

$$\sum_{j=0}^{\infty} |a_j|^p |a_j|^{-p} \Pr\{|\varepsilon_0| > |a_j|^{-1}\} \le \|\varepsilon_0\|_{p,\infty}^p \sum_{j=0}^{\infty} |a_j|^p. \tag{4.13}$$

Secondly, we investigate the convergence of series (4.11).

Suppose that $0 < p < 1$. Then

$$\sum_{j=0}^{\infty} |a_j| \operatorname{E}[|\varepsilon_0| I_{\{|\varepsilon_0| \le |a_j|^{-1}\}}] \le \sum_{j=0}^{\infty} |a_j| \|\varepsilon_0\|_{p,\infty}^p \frac{|a_j|^{p-1}}{1-p} = \frac{\|\varepsilon_0\|_{p,\infty}^p}{1-p} \sum_{j=0}^{\infty} |a_j|^p$$

using Lemma 4.2.

Suppose that $p = 1$. Then

$$\int_0^{|a_j|^{-1}} \Pr\{|\varepsilon_0| > s\} \mathrm{d}s \le 1 + \|\varepsilon_0\|_{1,\infty} \log |a_j|^{-1}$$

for $j \ge J$, where $J \ge 0$ is such that $|a_j|^{-1} \ge 1$ when $j \ge J$. By Lemma 4.2,

$$\sum_{j=J}^{\infty} |a_j| \operatorname{E}[|\varepsilon_0| I_{\{|\varepsilon_0| \le |a_j|^{-1}\}}] \le \sum_{j=J}^{\infty} |a_j| + \|\varepsilon_0\|_{1,\infty} \sum_{j=J}^{\infty} |a_j| \log |a_j|^{-1}.$$

Suppose that $p > 1$. We have that

$$\operatorname{E}[\varepsilon_0 I_{\{|\varepsilon_0| \le |a_j|^{-1}\}}] = \operatorname{E}[\varepsilon_0 I_{\{|\varepsilon_0| \le |a_j|^{-1}\}}] - \operatorname{E}\varepsilon_0 = -\operatorname{E}[\varepsilon_0 I_{\{|\varepsilon_0| > |a_j|^{-1}\}}] \tag{4.14}$$

and

$$\begin{aligned} \operatorname{E}[|\varepsilon_0| I_{\{|\varepsilon_0| > |a_j|^{-1}\}}] &= \int_0^{|a_j|^{-1}} \Pr\{|\varepsilon_0| I_{\{|\varepsilon_0| > |a_j|^{-1}\}} > x\} \mathrm{d}x \\ &\quad + \int_{|a_j|^{-1}}^{\infty} \Pr\{|\varepsilon_0| I_{\{|\varepsilon_0| > |a_j|^{-1}\}} > x\} \mathrm{d}x \\ &= |a_j|^{-1} \Pr\{|\varepsilon_0| > |a_j|^{-1}\} + \int_{|a_j|^{-1}}^{\infty} \Pr\{|\varepsilon_0| > x\} \mathrm{d}x \end{aligned}$$

since $\Pr\{|\varepsilon_0| I_{\{|\varepsilon_0|>|a_j|^{-1}\}} > x\} = \Pr\{|\varepsilon_0| > |a_j|^{-1}\}$ for $0 \le x \le |a_j|^{-1}$. Hence,

$$\sum_{j=0}^{\infty} |a_j| \operatorname{E}[|\varepsilon_0| I_{\{|\varepsilon_0|>|a_j|^{-1}\}}] \le \sum_{j=0}^{\infty} |a_j| \|\varepsilon_0\|_{p,\infty}^p \Big[1 + \frac{1}{p-1}\Big] |a_j|^{p-1}$$
$$= \|\varepsilon_0\|_{p,\infty}^p \Big[1 + \frac{1}{p-1}\Big] \sum_{j=0}^{\infty} |a_j|^p.$$

Finally, we complete the proof by establishing the convergence of series (4.12).

Suppose that $0 < p < 2$. Then

$$\sum_{j=0}^{\infty} |a_j|^2 \operatorname{E}[|\varepsilon_0|^2 I_{\{|\varepsilon_0|\le|a_j|^{-1}\}}] \le \sum_{j=0}^{\infty} |a_j|^2 2\|\varepsilon_0\|_{p,\infty}^p \frac{|a_j|^{p-2}}{2-p} = \frac{2\|\varepsilon_0\|_{p,\infty}^p}{2-p} \sum_{j=0}^{\infty} |a_j|^p$$

using Lemma 4.2.

Suppose that $p = 2$. Then

$$\int_0^{|a_j|^{-1}} 2s \Pr\{|\varepsilon_0| > s\} \mathrm{d}s \le 1 + 2\|\varepsilon_0\|_{2,\infty}^2 \log |a_j|^{-1}$$

for $j \ge J$, where $J \ge 0$ is such that $|a_j|^{-1} \ge 1$ when $j \ge J$. By Lemma 4.2,

$$\sum_{j=J}^{\infty} |a_j|^2 \operatorname{E}[|\varepsilon_0|^2 I_{\{|\varepsilon_0|\le|a_j|^{-1}\}}] \le \sum_{j=J}^{\infty} |a_j|^2 + 2\|\varepsilon_0\|_{2,\infty}^2 \sum_{j=J}^{\infty} |a_j|^2 \log |a_j|^{-1}.$$

Suppose that $p > 2$. Then $\operatorname{Var}[\varepsilon_0 I_{\{|\varepsilon_0|\le|a_j|^{-1}\}}] \to \operatorname{Var} \varepsilon_0$ as $j \to \infty$ and the series

$$\sum_{j=0}^{\infty} a_j^2 \operatorname{Var}[\varepsilon_0 I_{\{|\varepsilon_0|\le|a_j|^{-1}\}}]$$

converges if $\sum_{j=0}^{\infty} a_j^2 < \infty$. □

Remark 4.1. Suppose that $a_j > 0$ for each $j \ge 0$ and that ε_0 has the density function

$$f(x) = \begin{cases} \frac{2}{\pi} \frac{1}{1+x^2} & \text{if } x \ge 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then $\varepsilon_0 \in L_{1,\infty}$ and

$$\operatorname{E}[\varepsilon_0 I_{\{\varepsilon_0 \le a_j^{-1}\}}] = \frac{2}{\pi} \int_0^{a_j^{-1}} \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{\pi} \log(a_j^{-2} + 1)$$

so that the series

$$\sum_{j=0}^{\infty} \operatorname{E}[a_j \varepsilon_{k-j} I_{\{|a_j \varepsilon_{k-j}| \le 1\}}]$$

converges if and only if $\sum_{j=0}^{\infty} a_j \log a_j^{-1} < \infty$ since $\log(a_j^{-2}+1) \sim 2\log a_j^{-1}$. Hence, the condition in Proposition 4.3 when $p = 1$ is sharp. Similarly, we can construct an example to show that the condition in Proposition 4.3 when $p = 2$ is also sharp.

Proposition 4.4. *Suppose that* $\mathrm{E}\,|\varepsilon_0|^p < \infty$ *for* $p > 0$ *and* $\mathrm{E}\,\varepsilon_0 = 0$ *if* $p \geq 1$. *Series* (4.1) *converges almost surely if* $\sum_{j=0}^{\infty} |a_j|^q < \infty$, *where* $q = \min\{p, 2\}$.

Proof. We use Proposition 4.3 to establish the almost sure convergence of series (4.1) when $p \neq 1$ and $p \neq 2$ since $L_p \subset L_{p,\infty}$. In order to show that series (4.1) converges when $p = 1$ and $p = 2$, we again use Kolmogorov's three-series theorem as in the proof of Proposition 4.3.

The convergence of series (4.10) follows from inequality (4.13) in both cases.

Series (4.11) converges since

$$\sum_{j=0}^{\infty} |a_j|\,\mathrm{E}[|\varepsilon_0| I_{\{|\varepsilon_0| \leq |a_j|^{-1}\}}] \leq \mathrm{E}\,|\varepsilon_0| \sum_{j=0}^{\infty} |a_j|$$

when $p = 1$ and, using equation (4.14),

$$\sum_{j=0}^{\infty} |a_j|\,\mathrm{E}[|\varepsilon_0| I_{\{|\varepsilon_0| > |a_j|^{-1}\}}] = \sum_{j=0}^{\infty} |a_j|\,\mathrm{E}[|\varepsilon_0| \frac{|a_j|^{-1}}{|a_j|^{-1}} I_{\{|\varepsilon_0| > |a_j|^{-1}\}}] \leq \mathrm{E}\,\varepsilon_0^2 \sum_{j=0}^{\infty} a_j^2$$

when $p = 2$.

Series (4.12) converges since

$$\sum_{j=0}^{\infty} a_j^2\,\mathrm{E}[|\varepsilon_0|^2 I_{\{|\varepsilon_0| \leq |a_j|^{-1}\}}] \leq \sum_{j=0}^{\infty} |a_j|\,\mathrm{E}[|\varepsilon_0| I_{\{|\varepsilon_0| \leq |a_j|^{-1}\}}] \leq \mathrm{E}\,|\varepsilon_0| \sum_{j=0}^{\infty} |a_j|.$$

when $p = 1$ and $\mathrm{Var}[\varepsilon_0 I_{\{|\varepsilon_0| \leq |a_j|^{-1}\}}] \to \mathrm{Var}\,\varepsilon_0$ as $j \to \infty$ when $p = 2$. □

Remark 4.2. If $\mathrm{E}\,\varepsilon_0 \neq 0$ and $a_j \geq 0$ for each $j \geq 0$, then series (4.1) converges if and only if $\sum_{j=0}^{\infty} a_j < \infty$. If $\mathrm{E}\,\varepsilon_0 = 0$, $\mathrm{E}\,\varepsilon_0^2 < \infty$ and $\mathrm{E}\,\varepsilon_0^2 \neq 0$, then series (4.1) converges if and only if $\sum_{j=0}^{\infty} a_j^2 < \infty$.

$\mathbb{H}$-valued case

Proposition 4.5. *Suppose that* $\|\varepsilon_0\|_{p,\infty} < \infty$ *for* $1 < p < 2$ *and* $\mathrm{E}\,\varepsilon_0 = 0$. *Then series* (4.1) *converges almost surely if* $\sum_{j=0}^{\infty} \|a_j\|^p < \infty$.

Proof. Let $N > M$ and let $\{\tilde{\varepsilon}_j\} = \{\tilde{\varepsilon}_j : j \in \mathbb{Z}\}$ be an independent copy of $\{\varepsilon_j\}$ so that $\{\varepsilon_j - \tilde{\varepsilon}_j\} = \{\varepsilon_j - \tilde{\varepsilon}_j : j \in \mathbb{Z}\}$ are independent and identically distributed symmetric random elements. Then

$$\mathrm{E}\Big\| \sum_{j=M+1}^{N} a_j(\varepsilon_{k-j}) \Big\| \leq \mathrm{E}\Big\| \sum_{j=M+1}^{N} a_j(\varepsilon_{k-j} - \tilde{\varepsilon}_{k-j}) \Big\|$$

using the fact that $\mathrm{E}\,\tilde{\varepsilon}_0 = 0$.

We obtain that

$$\begin{aligned}
\mathrm{E}\Big\| \sum_{j=M+1}^{N} a_j(\varepsilon_{k-j} - \tilde{\varepsilon}_{k-j}) \Big\| &\leq \Big(\frac{p}{p-1}\Big) \Big\| \sum_{j=M+1}^{N} a_j(\varepsilon_{k-j} - \tilde{\varepsilon}_{k-j}) \Big\|_{p,\infty} \\
&\leq C^{1/p}\Big(\frac{p}{p-1}\Big)\Big(\sum_{j=M+1}^{N} \|a_j(\varepsilon_0 - \tilde{\varepsilon}_0)\|_{p,\infty}^p \Big)^{1/p} \\
&\leq C^{1/p}\Big(\frac{p}{p-1}\Big)\Big(\sum_{j=M+1}^{N} \| \|a_j\| (\varepsilon_0 - \tilde{\varepsilon}_0)\|_{p,\infty}^p \Big)^{1/p} \\
&\leq 4C^{1/p}\Big(\frac{p}{p-1}\Big) \|\varepsilon_0\|_{p,\infty} \Big(\sum_{j=M+1}^{N} \|a_j\|^p \Big)^{1/p}
\end{aligned}$$

using the fact that $1 < p < 2$, inequalities (4.8) and (4.9), the inequality

$$\|a_j(\varepsilon_0 - \tilde{\varepsilon}_0)\| \leq \|a_j\| \|\varepsilon_0 - \tilde{\varepsilon}_0\|$$

for each $j \geq 0$ and inequality (4.7).

The convergence in mean of series (4.1) implies that it converges in probability and since the random elements $\{\varepsilon_k : k \in \mathbb{Z}\}$ are independent for each $k \geq 1$, it also follows that the series converges almost surely (see Theorem 6.1 of Ledoux and Talagrand [35]). The proof is complete. □

Using the fact that $L_p \subset L_{p,\infty}$, we obtain the following corollary of Proposition 4.5.

Corollary 4.2. *Suppose that* $\|\varepsilon_0\|_p < \infty$ *for* $1 < p < 2$ *and* $\mathrm{E}\,\varepsilon_0 = 0$. *Then series* (4.1) *converges almost surely if* $\sum_{j=0}^{\infty} \|a_j\|^p < \infty$.

4.2.3 Symmetrization

Some lemmas and inequalities that we use require symmetric random elements (for example, inequality (4.9) above and Lemma 4.5 below). We need to use symmetrization to establish results for not necessarily symmetric random elements.

Suppose that $\{\xi_k\} = \{\xi_k : k \in \mathbb{Z}\}$ are independent $\mathbb{H}$-valued random elements. We construct an associated sequence of independent and symmetric random elements by setting, for each $k \in \mathbb{Z}$, $\xi_k - \tilde{\xi}_k$, where $\{\tilde{\xi}_k : k \in \mathbb{Z}\}$ is an independent copy of a sequence $\{\xi_k\}$. If $\mathrm{E}\,\xi_0 = 0$, then we have the following inequality

$$\mathrm{E}\,\|\xi_0\| = \mathrm{E}\,\|\xi_0 - \mathrm{E}\,\tilde{\xi}_0\| \leq \mathrm{E}\,\|\xi_0 - \tilde{\xi}_0\|$$

since $\mathrm{E}\,\tilde{\xi}_0 = 0$. Hence, we can estimate from above the first moment of a not necessarily symmetric random element with the first moment of a symmetrized random element.

The following lemma ensures that we can deduce the strong law of large numbers for not necessarily symmetric random elemnents from the strong law of large numbers for the symmetrized sequence (see Lemma 7.1 of Ledoux and Talagrand [35]).

Lemma 4.3. *Let $\{\xi_n\} = \{\xi_n : n \geq 1\}$ and $\{\xi'_n\} = \{\xi'_n : n \geq 1\}$ be independent sequences of $\mathbb{H}$-valued random elements such that the sequence $\{\xi_n - \xi'_n\}$ is almost surely convergent to 0 as $n \to \infty$ and $\{\xi_n\}$ is convergent to 0 in probability as $n \to \infty$. Then $\{\xi_n\}$ is almost surely convergent to 0.*

So we assume without loss of generality that $\{\varepsilon_k\}$ are symmetric random variables in the proofs of Theorem 4.2 and Theorem 4.4.

4.2.4 Auxiliary lemmas

Suppose that $\mathbb{E}$ is a separable Banach space and $\{a_j\} = \{a_j : j \geq 0\}$ are bounded linear operators from $\mathbb{E}$ to $\mathbb{E}$ and suppose that $\{U_{nj}\} = \{U_{nj} : n \geq 1, j \geq 0\}$ are random elements with values in $\mathbb{E}$.

Lemma 4.4. *Suppose that $\sum_{j=0}^{\infty} \|a_j\| < \infty$. Then $A = \sum_{j=0}^{\infty} a_j$ is a bounded linear operator from $\mathbb{E}$ to $\mathbb{E}$. If*

$$\sup_{n\geq 1, j\geq 0} \mathrm{E}\,\|U_{nj}\| < \infty,$$

then the series

$$\sum_{j=0}^{\infty} a_j(U_{nj})$$

converges almost surely for each $n \geq 1$ and if, in addition,

$$\|U_{ni} - U_{nj}\| \to 0$$

in probability as $n \to \infty$ for each $i \geq 0$ and $j \geq 0$, then

$$\left\| \sum_{j=0}^{\infty} a_j(U_{nj}) - A(U_{ni}) \right\| \to 0$$

in probability as $n \to \infty$ for each $i \geq 0$.

See Račkauskas and Suquet [47], for the proof of Lemma 4.4.

Lemma 4.5. *Suppose that $\{\xi_k\} = \{\xi_k : k \geq 1\}$ are independent and symmetric random elements of a separable Banach space $\mathbb{E}$ and $\{b_n\} = \{b_n : n \geq 1\}$ is a sequence of positive numbers. Let $S_k = \sum_{j=1}^{k} \xi_j$ for each $k \geq 1$. Then*

$$\Pr\bigl\{\sup_{k \geq n} \|b_k^{-1} S_k\| > \delta\bigr\} \leq 8 \sum_{k=n}^{\infty} k^{-1} \Pr\{\|b_k^{-1} S_k\| > \delta\}$$

for each $n \geq 1$ and for each $\delta > 0$.

The proof of Lemma 4.5 is similar to the part of the proof of Theorem 1.5 of Norvaiša and Račkauskas [44].

Proof. Let $i \geq 1$ be such that $2^{i-1} \leq n < 2^i$. We have that

$$\begin{aligned}
\Pr\left\{ \sup_{k \geq n} \|b_k^{-1} S_k\| > \delta \right\} &\leq \Pr\left\{ \sup_{k \geq 2^{i-1}} \|b_k^{-1} S_k\| > \delta \right\} \\
&= \Pr\left\{ \sup_{j \geq i} \max_{2^{j-1} \leq k < 2^j} \|b_k^{-1} S_k\| > \delta \right\} \\
&= \Pr\left(\bigcup_{j \geq i} \left\{ \max_{2^{j-1} \leq k < 2^j} \|b_k^{-1} S_k\| > \delta \right\} \right) \\
&\leq \sum_{j=i}^{\infty} \Pr\left\{ \max_{2^{j-1} \leq k < 2^j} \|b_k^{-1} S_k\| > \delta \right\}
\end{aligned}$$

using the fact that

$$\left\{ \sup_{j \geq i} \max_{2^{j-1} \leq k < 2^j} \|b_k^{-1} S_k\| > \delta \right\} = \bigcup_{j \geq i} \left\{ \max_{2^{j-1} \leq k < 2^j} \|b_k^{-1} S_k\| > \delta \right\}$$

and subadditivity of the probability measure. By Lévy's inequality (see Kahane [28], p. 14),

$$\Pr\left\{ \max_{2^{j-1} \leq k < 2^j} \|b_k^{-1} S_k\| > \delta \right\} \leq 2 \Pr\{\|b_{2^j}^{-1} S_{2^j}\| > \delta\}.$$

We obtain the following inequalities

$$\Pr\left\{ \sup_{k \geq n} \|b_k^{-1} S_k\| > \delta \right\} \leq 2 \sum_{j=i}^{\infty} \Pr\{\|b_{2^j}^{-1} S_{2^j}\| > \delta\}$$

$$
\begin{aligned}
&= 4\sum_{j=i}^{\infty} 2^{-j-1} \sum_{k=2^j}^{2^{j+1}-1} \Pr\{\|b_{2^j}^{-1} S_{2^j}\| > \delta\} \\
&\le 4\sum_{j=i}^{\infty} \sum_{k=2^j}^{2^{j+1}-1} k^{-1} \Pr\{\max_{2^j \le l \le k} \|b_l^{-1} S_l\| > \delta\} \\
&\le 8\sum_{k=2^j}^{\infty} k^{-1} \Pr\{\|b_k^{-1} S_k\| > \delta\} \\
&\le 8\sum_{k=n}^{\infty} k^{-1} \Pr\{\|b_k^{-1} S_k\| > \delta\}.
\end{aligned}
$$

The proof is complete. □

Lemma 4.6. *Let $q > p$ and $c > 0$. Then*

$$
\sum_{k=1}^{\infty} \frac{1}{k^{q/p}} \operatorname{E}[\|\varepsilon_0\|^q I_{\{\|\varepsilon_0\| \le ck^{1/p}\}}] \le C \cdot c^{q-p} \operatorname{E} \|\varepsilon_0\|^p,
$$

where C is a positive constant.

The proof of Lemma 4.6 is essentially similar to the proof of Lemma 6.1 of Section 6.6 of Gut [22].

Proof. We have that

$$
\begin{aligned}
&\sum_{k=1}^{\infty} \frac{1}{k^{q/p}} \operatorname{E}[\|\varepsilon_0\|^q I_{\{\|\varepsilon_0\| \le ck^{1/p}\}}] = \\
&\quad = \sum_{k=1}^{\infty} \frac{1}{k^{q/p}} \sum_{l=1}^{k} \operatorname{E}[\|\varepsilon_0\|^q I_{\{c(l-1)^{1/p} < \|\varepsilon_0\| \le cl^{1/p}\}}] \\
&\quad = \sum_{l=1}^{\infty} \sum_{k=l}^{\infty} \frac{1}{k^{q/p}} \operatorname{E}[\|\varepsilon_0\|^q I_{\{c(l-1)^{1/p} < \|\varepsilon_0\| \le cl^{1/p}\}}] \\
&\quad = \left(\sum_{k=1}^{\infty} \frac{1}{k^{q/p}}\right) \operatorname{E}[\|\varepsilon_0\|^q I_{\{\|\varepsilon_0\| \le c\}}] + \sum_{l=2}^{\infty} \left(\sum_{k=l}^{\infty} \frac{1}{k^{q/p}}\right) \operatorname{E}[\|\varepsilon_0\|^q I_{\{c(l-1)^{1/p} < \|\varepsilon_0\| \le cl^{1/p}\}}].
\end{aligned}
$$

We obtain

$$
\operatorname{E}[\|\varepsilon_0\|^q I_{\{\|\varepsilon_0\| \le c\}}] \le c^{q-p} \operatorname{E}[\|\varepsilon_0\|^p I_{\{\|\varepsilon_0\| \le c\}}] \le c^{q-p} \operatorname{E} \|\varepsilon_0\|^p
$$

and

$$
\begin{aligned}
&\sum_{l=2}^{\infty} \left(\sum_{k=l}^{\infty} \frac{1}{k^{q/p}}\right) \operatorname{E}[\|\varepsilon_0\|^q I_{\{c(l-1)^{1/p} < \|\varepsilon_0\| \le cl^{1/p}\}}] \le \\
&\quad \le \frac{2^{q/p-1}}{q/p-1} \sum_{l=2}^{\infty} \frac{1}{l^{q/p-1}} \operatorname{E}[\|\varepsilon_0\|^q I_{\{c(l-1)^{1/p} < \|\varepsilon_0\| \le cl^{1/p}\}}]
\end{aligned}
$$

$$\leq \frac{2^{q/p-1}c^{q-p}}{q/p-1}\sum_{l=2}^{\infty}\mathrm{E}[\|\varepsilon_0\|^p I_{\{c(l-1)^{1/p}<\|\varepsilon_0\|\leq cl^{1/p}\}}]$$
$$\leq \frac{2^{q/p-1}c^{q-p}}{q/p-1}\,\mathrm{E}\,\|\varepsilon_0\|^p.$$

The proof is complete. □

4.3 Proofs

We make use of the technique of truncation. Let us introduce several notations. Suppose that $\{r_{nj}\} = \{r_{nj} : n \geq 1, j \in \mathbb{Z}\}$ are positive real numbers. Denote

$$\mu'_{nj} = \mathrm{E}\big[\varepsilon_0 I_{\{\|\varepsilon_0\|\leq r_{nj}\}}\big] \quad \text{and} \quad \mu''_{nj} = \mathrm{E}\big[\varepsilon_0 I_{\{\|\varepsilon_0\|> r_{nj}\}}\big], \tag{4.15}$$

for $n \geq 1$ and $j \in \mathbb{Z}$. Set $\varepsilon_j = \varepsilon'_{nj} + \varepsilon''_{nj}$, where

$$\varepsilon'_{nj} = \varepsilon_j I_{\{\|\varepsilon_j\|\leq r_{nj}\}} - \mu'_{nj} \quad \text{and} \quad \varepsilon''_{nj} = \varepsilon_j I_{\{\|\varepsilon_j\|> r_{nj}\}} - \mu''_{nj}$$

so that $\mathrm{E}\,\varepsilon_0 = \mathrm{E}\,\varepsilon'_{nj} = \mathrm{E}\,\varepsilon''_{nj} = 0$ for $n \geq 1$ and $j \in \mathbb{Z}$. Denote

$$S'_n = \sum_{j=-\infty}^{n} w_{nj}\varepsilon'_{nj} \quad \text{and} \quad S''_n = \sum_{j=-\infty}^{n} w_{nj}\varepsilon''_{nj} \tag{4.16}$$

for $n \geq 1$.

We have that

$$\mathrm{E}\,\|\varepsilon'_{nj}\|^2 = \mathrm{E}\big[\|\varepsilon_0\|^2 I_{\{\|\varepsilon_0\|\leq r_{nj}\}}\big] - \|\mu'_{nj}\|^2 \tag{4.17}$$

using the fact that $\|x\| = \sqrt{\langle x, x\rangle}$ for $x \in \mathbb{H}$ and the fact that the expectation commutes with the bounded operators.

By Lemma 4.2,

$$\mathrm{E}\big[\|\varepsilon_0\|^q I_{\{\|\varepsilon_0\|\leq r_{nj}\}}\big] \leq \int_0^{r_{nj}} qs^{q-1}\,\mathrm{Pr}\{\|\varepsilon_0\| > s\}\mathrm{d}s \leq \frac{q}{q-p}\|\varepsilon_0\|^p_{p,\infty}\cdot r_{nj}^{q-p} \tag{4.18}$$

for $q > p > 0$.

4.3.1 Summable linear filter

We have that

$$S_n = \sum_{k=1}^{n} X_k = \sum_{k=1}^{n}\sum_{j=0}^{\infty} a_j(\varepsilon_{k-j}) = \sum_{j=0}^{\infty} a_j(s_{nj})$$

using additivity of the bounded linear operators $\{a_j\}$, where

$$s_{nj} = \sum_{k=1}^{n} \varepsilon_{k-j}. \tag{4.19}$$

Notice that $\{s_{nj} : j \geq 0\}$ are not independent for any $n > 1$. Denote

$$\mathcal{A} = \sum_{j=0}^{\infty} \|a_j\|.$$

Using the monotone convergence theorem, the inequality $\|a_j(s_{nj})\| \leq \|a_j\|\|s_{nj}\|$ for each $n \geq 1$ and each $j \geq 0$ and the triangle inequality, we obtain

$$\mathrm{E}\Bigg[\sum_{j=0}^{\infty} \|a_j(s_{nj})\|\Bigg] = \sum_{j=0}^{\infty} \mathrm{E}\,\|a_j(s_{nj})\| \leq \mathcal{A}\,\mathrm{E}\,\|s_{n0}\| \leq \mathcal{A}n\,\mathrm{E}\,\|\varepsilon_0\|$$

for $n \geq 1$. Hence

$$\mathrm{E}\Bigg\|\sum_{j=0}^{\infty} a_j(s_{nj})\Bigg\| = \lim_{m\to\infty} \mathrm{E}\Bigg\|\sum_{j=0}^{m} a_j(s_{nj})\Bigg\| \tag{4.20}$$

using the dominated convergence theorem.

Using (4.20), the triangle inequality, Hölder's inequality and the von Bahr-Esseen inequality (see von Bahr and Esseen [56]), we obtain

$$\mathrm{E}\,\|S_n\| \leq \mathcal{A}\,\mathrm{E}\,\|s_{n0}\| \leq \mathcal{A}(\mathrm{E}\,\|s_{n0}\|^p)^{1/p} \leq 2^{1/p}\mathcal{A}(\mathrm{E}\,\|\varepsilon_0\|^p)^{1/p} \cdot n^{1/p} \tag{4.21}$$

for $1 \leq p \leq 2$. If $p = 2$, $2^{1/p} = 2^{1/2}$ can by replaced by 1 in inequality (4.21).

Weak law of large numbers

We have that

$$\Pr\{\|n^{-1/p}S_n\| > \delta\} \leq \Pr\{\|n^{-1/p}S'_n\| > \delta/2\} + \Pr\{\|n^{-1/p}S''_n\| > \delta/2\} \tag{4.22}$$

for each $\delta > 0$.

Lemma 4.7. *Suppose that $x^p \Pr\{\|\varepsilon_0\| > x\} \to 0$ as $x \to \infty$ for some $p > 1$ and $r_n \to \infty$ as $n \to \infty$, where $r_n = \inf_{j\leq n} r_{nj}$. Then*

$$\sup_{j\leq n} \|\varepsilon''_{nj}\|_{p,\infty} \to 0 \quad \textit{as} \quad n \to \infty.$$

Proof. Denote

$$M''_n = \mathrm{E}[\|\varepsilon_0\| I_{\{\|\varepsilon_0\|>r_n\}}]$$

and observe that

$$\|\mu''_{nj}\| \leq \mathrm{E}\big[\|\varepsilon_0\| I_{\{\|\varepsilon_0\|>r_{nj}\}}\big] \leq M''_n \tag{4.23}$$

since $r_{nj} \geq r_n$.

Using triangle inequality and inequality (4.23),

$$\begin{aligned}\sup_{j\leq n} \|\varepsilon''_{nj}\|^p_{p,\infty} &= \sup_{j\leq n}\sup_{x>0}(x^p \Pr\{\|\varepsilon_0 I_{\{\|\varepsilon_0\|>r_{nj}\}} - \mu''_{nj}\| > x\}) \\ &\leq \sup_{j\leq n}\sup_{x>0}(x^p \Pr\{\|\varepsilon_0\| I_{\{\|\varepsilon_0\|>r_{nj}\}} + \|\mu''_{nj}\| > x\}) \\ &\leq \sup_{x>0}(x^p \Pr\{\|\varepsilon_0\| I_{\{\|\varepsilon_0\|>r_n\}} > x - M''_n\}).\end{aligned}$$

Observe that

$$\Pr\{\|\varepsilon_0\| I_{\{\|\varepsilon_0\|>r_n\}} > x - M''_n\} = \begin{cases} 1, & \text{if } 0 < x < M''_n, \\ \Pr\{\|\varepsilon_0\| > r_n\}, & \text{if } M''_n \leq x \leq M''_n + r_n, \\ \Pr\{\|\varepsilon_0\| > x\}, & \text{if } x > M''_n + r_n. \end{cases}$$

We obtain

$$\begin{aligned}&\sup_{x>0}(x^p \Pr\{\|\varepsilon_0\| I_{\{\|\varepsilon_0\|>r_n\}} > x - M''_n\}) \leq \\ &\quad \leq \max\{(M''_n)^p, (M''_n + r_n)^p \Pr\{\|\varepsilon_0\| > r_n\}, \sup_{x>M''_n+r_n}(x^p \Pr\{\|\varepsilon_0\| > x\})\} \to 0\end{aligned}$$

as $n \to \infty$ since $x^p \Pr\{\|\varepsilon_0\| > x\} \to 0$ as $x \to \infty$ and $r_n \to \infty$ as $n \to \infty$, so that $\lim_{n\to\infty} M''_n = 0$. The proof is complete. □

Proof of Theorem 4.1. For $j \geq 0$, $n \geq 1$ and $\tau > 0$, set

$$r_{nj} = \left[\tau^2 \cdot \left(\frac{\delta}{2\mathcal{A}}\right)^2 \cdot \frac{2-p}{2\|\varepsilon_0\|^p_{p,\infty}}\right]^{\frac{1}{2-p}} n^{1/p}.$$

Using Markov's inequality, inequalities (4.21), (4.17) and (4.18),

$$\Pr\{\|n^{-1/p} S'_n\| > \delta/2\} \leq \frac{2}{\delta}\frac{\mathrm{E}\,\|S'_n\|}{n^{1/p}} \leq \frac{2}{\delta}\mathcal{A}\frac{(\mathrm{E}\,\|\varepsilon'_{n0}\|^2)^{1/2}}{n^{1/p-1/2}} \leq \tau.$$

Hence, the first term on the right side of (4.22) goes to 0 as $n \to \infty$ for each $\delta > 0$.

Let $\{\tilde{\varepsilon}''_{nj} : n \geq 1, j \in \mathbb{Z}\}$ be an idependent copy of $\{\varepsilon''_{nj} : n \geq 1, j \in \mathbb{Z}\}$ so that $\{\varepsilon''_{nj} - \tilde{\varepsilon}''_{nj} : n \geq 1, j \in \mathbb{Z}\}$ are independent and symmetric $\mathbb{H}$-valued random elements. Let us observe that

$$\mathrm{E}\,\|s''_{n0}\| = \mathrm{E}\left\|\sum_{k=1}^{n} \varepsilon''_{nk}\right\| = \mathrm{E}\left\|\sum_{k=1}^{n} (\varepsilon''_{nk} - \mathrm{E}\,\tilde{\varepsilon}''_{nk})\right\| \leq \mathrm{E}\left\|\sum_{k=1}^{n} (\varepsilon''_{nk} - \tilde{\varepsilon}''_{nk})\right\| \tag{4.24}$$

since $\mathrm{E}\,\tilde{\varepsilon}''_{nj} = 0$ for each $n \geq 1$ and for each $j \in \mathbb{Z}$.

Using Markov's inequality, inequalities (4.21), (4.24), (4.8), (4.9) and (4.7), we obtain the following inequalities

$$\begin{aligned}
\Pr\{\|n^{-1/p} S''_n\| > \delta/2\} &\leq \frac{2}{\delta} n^{-1/p}\,\mathrm{E}\,\|S''_n\| \\
&\leq \frac{2}{\delta}\mathcal{A} n^{-1/p}\,\mathrm{E}\,\|s''_{n0}\| \\
&\leq \frac{2}{\delta}\mathcal{A} n^{-1/p}\,\mathrm{E}\left\|\sum_{k=1}^{n} (\varepsilon''_{nk} - \tilde{\varepsilon}''_{nk})\right\| \\
&\leq \frac{2}{\delta}\mathcal{A} n^{-1/p}\left(\frac{p}{p-1}\right)\left\|\sum_{k=1}^{n} (\varepsilon''_{nk} - \tilde{\varepsilon}''_{nk})\right\|_{p,\infty} \\
&\leq \frac{2}{\delta}\mathcal{A} n^{-1/p}\left(\frac{p}{p-1}\right) C^{1/p}\left(\sum_{k=1}^{n} \|\varepsilon''_{n0} - \tilde{\varepsilon}''_{n0}\|_{p,\infty}^{p}\right)^{1/p} \\
&\leq \frac{8}{\delta}\mathcal{A}\left(\frac{p}{p-1}\right) C^{1/p}\|\varepsilon''_{n0}\|_{p,\infty}
\end{aligned}$$

$\|\varepsilon''_{n0}\|_{p,\infty} \to 0$ as $n \to \infty$ using Lemma 4.7 since $x^p \Pr\{|\varepsilon_0| > x\} \to 0$ as $x \to \infty$ and $r_n \to \infty$ as $n \to \infty$. The proof is complete. □

Theorem 4.1 can also be proved using Lemma 4.4.

Proof of Theorem 4.1. Set

$$U_{nj} = n^{-1/p} s_{nj},$$

where $\{s_{nj}\} = \{s_{nj} : n \geq 1, j \geq 0\}$ is defined by (4.19) for each $n \geq 1$ and $j \geq 0$.

First, we have that

$$\begin{aligned}
\sup_{n\geq 1, j\geq 0} \mathrm{E}\,\|n^{-1/p} s_{nj}\| &\leq \sup_{n\geq 1, j\geq 0} \mathrm{E}\left\|n^{-1/p}\sum_{k=1}^{n} (\varepsilon_{k-j} - \tilde{\varepsilon}_{k-j})\right\| \\
&\leq \left(\frac{p}{p-1}\right) \sup_{n\geq 1, j\geq 0}\left\|n^{-1/p}\sum_{k=1}^{n} (\varepsilon_{k-j} - \tilde{\varepsilon}_{k-j})\right\|_{p,\infty} \\
&\leq \left(\frac{p}{p-1}\right) \sup_{n\geq 1, j\geq 0}\left(n^{-1}\left\|\sum_{k=1}^{n} (\varepsilon_{k-j} - \tilde{\varepsilon}_{k-j})\right\|_{p,\infty}^{p}\right)^{1/p}
\end{aligned}$$

$$\leq 4\left(\frac{p}{p-1}\right)C^{1/p}\|\varepsilon_0\|_{p,\infty}$$

using inequalities (4.8) and (4.9) since $p > 1$, where $\{\tilde{\varepsilon}_k : k \in \mathbb{Z}\}$ is an idependent copy of $\{\varepsilon_k : k \in \mathbb{Z}\}$ so that $\{\varepsilon_k - \tilde{\varepsilon}_k : k \in \mathbb{Z}\}$ are independent and identically distributed symmetric $\mathbb{H}$-valued random elements such that

$$\mathrm{E}\,\|s_{nj}\| = \mathrm{E}\left\|\sum_{k=1}^{n}\varepsilon_{k-j}\right\| = \mathrm{E}\left\|\sum_{k=1}^{n}(\varepsilon_{k-j} - \mathrm{E}\,\tilde{\varepsilon}_{k-j})\right\| \leq \mathrm{E}\left\|\sum_{k=1}^{n}(\varepsilon_{k-j} - \tilde{\varepsilon}_{k-j})\right\|$$

since $\mathrm{E}\,\tilde{\varepsilon}_k = 0$ for each $k \in \mathbb{Z}$.

Secondly,

$$\|U_{ni} - U_{nj}\| = \|n^{-1/p}s_{ni} - n^{-1/p}s_{nj}\| \to 0$$

in probability as $n \to \infty$ for each $i \geq 0$ and for each $j \geq 0$ using the Marcinkiewicz-Zygmund weak law of large numbers.

Finally, we have that

$$\left\|\sum_{j=0}^{\infty} a_j(n^{-1/p}s_{nj}) - A(n^{-1/p}s_{n0})\right\| = \|n^{-1/p}S_n - A(n^{-1/p}s_{n0})\| \to 0$$

in probability as $n \to \infty$ using Lemma 4.4. Hence, $n^{-1/p}S_n \to 0$ in probability as $n \to \infty$ and the proof is complete. □

Srong law of large numbers

Proof of Theorem 4.2. The proof is based on Lemma 4.4 and the fact that

$$n^{-1/p}S_n \to 0$$

almost surely as $n \to \infty$ if and only if

$$\sup_{k \geq n}\|k^{-1/p}S_k\| \to 0$$

in probability as $n \to \infty$.

Let $c_0(\mathbb{H})$ be a separable Banach space of $\mathbb{H}$-valued sequences that converge to 0 with the norm given by

$$\|x\| = \sup_{n \geq 1}\|x_n\|$$

for each $x \in c_0(\mathbb{H})$.

Let $\{u_j\} = \{u_j : j \geq 0\}$ be bounded linear operators from $c_0(\mathbb{H})$ to $c_0(\mathbb{H})$ defined by

$$u_j(x) = \{a_j x_k : k \geq 1\}$$

for each $x \in c_0(\mathbb{H})$, where $\{a_j\} = \{a_j : j \geq 0\}$ are bounded linear operators from $\mathbb{H}$ to $\mathbb{H}$. Then we have that

$$\|u_j\| = \sup_{\|x\|\leq 1} \|u_j(x)\| = \sup_{\|x\|\leq 1} \sup_{n\geq 1} \|a_j x_n\| \leq \sup_{\|x\|\leq 1} \sup_{n\geq 1} \|a_j\|\|x_n\| = \|a_j\|.$$

Hence, $\sum_{j=0}^{\infty} \|u_j\| < \infty$ since $\sum_{j=0}^{\infty} \|a_j\| < \infty$.

Set

$$U_{nj} = \{0, \ldots, 0, n^{-1/p} s_{nj}, (n+1)^{-1/p} s_{(n+1)j}, \ldots\}$$

for each $n \geq 1$ and for each $j \geq 0$. Since $n^{1/p} s_{nj} \to 0$ almost surely as $n \to \infty$ for each $j \geq 0$ using the Marcinkiewicz–Zygmund strong law of large numbers, we have that U_{nj} almost surely belongs to $c_0(\mathbb{H})$ for each $n \geq 1$ and $j \geq 0$.

Now we show that $\sup_{n\geq 1, j\geq 0} \mathrm{E}\,\|U_{nj}\|$ is finite.

Lemma 4.8. *If* $\mathrm{E}\,\|\varepsilon_0\|^p < \infty$ *for some* $p > 1$*, then*

$$\sup_{n\geq 1, j\geq 0} \mathrm{E}\,\|U_{nj}\| < \infty.$$

Proof. We will use the inequality

$$\sum_{i=1}^{\infty} \Pr\{\|\xi\| \geq i^{1/r}\} \leq \mathrm{E}\,\|\xi\|^r \leq \sum_{i=0}^{\infty} \Pr\{\|\xi\| > i^{1/r}\}, \tag{4.25}$$

where ξ is a random element and $r > 0$ (see Chow and Teicher [8], p. 90).

Using Lemma 4.5,

$$\Pr\Big\{\sup_{k\geq n} \|k^{-1/p} s_{kj}\| > i\Big\} \leq 8 \sum_{k=n}^{\infty} k^{-1} \Pr\{\|k^{-1/p} s_{kj}\| > i\}$$

for each $n \geq 1$ and $i \geq 1$.

Set $r_{kj} = ik^{1/p}$ for each $k \geq 1$, each $j \geq 0$ and each $i \geq 1$. Using the truncated Chebyshev inequality (see Gut [22], p. 121), we obtain

$$\Pr\{\|k^{-1/p} s_{kj}\| > i\} \leq k \Pr\{\|\varepsilon_0\| > ik^{1/p}\} + i^{-2} k^{1-2/p}\,\mathrm{E}\,\|\varepsilon'_{k0}\|^2.$$

We have that

$$\sum_{k=n}^{\infty} \Pr\{\|\varepsilon_0\| > ik^{1/p}\} \le \mathrm{E}\,\|\varepsilon_0\|^p \cdot i^{-p}$$

and

$$i^{-2} \sum_{k=n}^{\infty} k^{-2/p}\,\mathrm{E}\,\|\varepsilon'_{k0}\|^2 \le C\,\mathrm{E}\,\|\varepsilon_0\|^p \cdot i^{-p}$$

using Lemma 4.6, where C is a positive constant.

Thus

$$\begin{aligned}\sum_{i=1}^{\infty} \Pr\Big\{\sup_{k\ge n} \|k^{-1/p} s_{kj}\| > i\Big\} &\le 8 \sum_{i=1}^{\infty}\sum_{k=n}^{\infty} k^{-1} \Pr\{\|k^{-1/p} s_{kj}\| > i\} \\ &\le 8 \sum_{i=1}^{\infty}\sum_{k=n}^{\infty} [\Pr\{\|\varepsilon_0\| > ik^{1/p}\} + i^{-2}k^{-2/p}\,\mathrm{E}\,\|\varepsilon'_{k0}\|^2] \\ &\le 8(1+C)\,\mathrm{E}\,\|\varepsilon_0\|^p \sum_{i=1}^{\infty} i^{-p}\end{aligned}$$

and

$$\sup_{n\ge1, j\ge0} \mathrm{E}\,\|U_{nj}\| = \sup_{n\ge1, j\ge0} \mathrm{E} \sup_{k\ge n} \|k^{-1/p} s_{kj}\| < \infty$$

since $p > 1$ and $\mathrm{E}\,\|\varepsilon_0\|^p < \infty$. The proof is complete. □

We also have that

$$\|U_{ni} - U_{nj}\| \to 0$$

in probability as $n \to \infty$ for each $i \ge 0$ and each $j \ge 0$ since

$$\sup_{k\ge n} \|k^{-1/p} s_{kj}\| \to 0$$

in probability as $n \to \infty$ as a consequence of the Marcinkiewicz-Zygmund strong law of large numbers. Hence, Lemma 4.4 implies that

$$\left\|\sum_{j=0}^{\infty} u_j(U_{nj}) - \sum_{j=0}^{\infty} u_j(U_{n0})\right\| \to 0$$

in probability as $n \to \infty$ and it follows using the reverse triangle inequality that

$$\left\|\sum_{j=0}^{\infty} u_j(U_{nj})\right\| = \sup_{k\ge n} \|k^{-1/p} S_k\| \to 0$$

in probability as $n \to \infty$. The proof of Theorem 4.2 is complete. □

4.3.2 Non-summable linear filter

Weak law of large numbers

We have that

$$\Pr\{\|b_n^{-1}(p)S_n\| > \delta\} \le \Pr\{\|b_n^{-1}(p)S_n'\| > \delta/2\} + \Pr\{\|b_n^{-1}(p)S_n''\| > \delta/2\} \quad (4.26)$$

for each $\delta > 0$.

Proposition 4.6. *Let $1 < p < 2$. If $x^p \Pr\{\|\varepsilon_0\| > x\} \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} r_n = \infty$, where $r_n = \inf_{j\le n} r_{nj}$, then*

$$b_n^{-1}(p)S_n'' \to 0$$

in probability as $n \to \infty$, where $b_n(p)$ is given by (4.3) *and S_n'' is given by* (4.16).

Proof. Let $\delta > 0$. For $N < n$, we have that

$$\Pr\{\|b_n^{-1}(p)S_n''\| > \delta/2\} \le \Pr\left\{\left\|b_n^{-1}(p)\sum_{j=-\infty}^{N} w_{nj}\varepsilon_j''\right\| > \delta/4\right\} + \Pr\left\{\left\|b_n^{-1}(p)\sum_{j=N+1}^{n} w_{nj}\varepsilon_j''\right\| > \delta/4\right\}. \quad (4.27)$$

The series $\sum_{j=-\infty}^{n} w_{nj}\varepsilon_j''$ converges almost surely for each $n \ge 1$. Therefore

$$b_n^{-1}(p)\sum_{j=-\infty}^{N} w_{nj}\varepsilon_j'' \to 0$$

almost surely as $N \to -\infty$ for each $n \ge 1$, so that there exists $N(n) < n$ for each $n \ge 1$ and each $\delta > 0$ such that

$$\left\|b_n^{-1}(p)\sum_{j=-\infty}^{N(n)} w_{nj}\varepsilon_j''\right\| \le \delta/4$$

almost surely and the first term on the right side of (4.27) is 0.

Let $\{\tilde{\varepsilon}_{nj}'' : n \ge 1,\ j \in \mathbb{Z}\}$ be an idependent copy of $\{\varepsilon_{nj}'' : n \ge 1,\ j \in \mathbb{Z}\}$ so that $\{\varepsilon_j'' - \tilde{\varepsilon}_j'' : n \ge 1,\ j \in \mathbb{Z}\}$ are independent and symmetric random variables.

Using Markov's inequality and the fact that $\operatorname{E}\tilde{\varepsilon}''_{nj} = 0$ for each $n \geq 1$ and each $j \in \mathbb{Z}$, we obtain

$$\Pr\Big\{\Big\|b_n^{-1}(p)\sum_{j=N(n)+1}^{n} w_{nj}\varepsilon''_j\Big\| > \delta/4\Big\} < \frac{4}{\delta}\operatorname{E}\Big\|b_n^{-1}(p)\sum_{j=N(n)+1}^{n} w_{nj}(\varepsilon''_j - \operatorname{E}\tilde{\varepsilon}''_{nj})\Big\|$$
$$\leq \frac{4}{\delta}\operatorname{E}\Big\|b_n^{-1}(p)\sum_{j=N(n)+1}^{n} w_{nj}(\varepsilon''_j - \tilde{\varepsilon}''_{nj})\Big\|$$

By inequalities (4.8),(4.9) and (4.7),

$$\Pr\Big\{\Big\|b_n^{-1}(p)\sum_{j=N(n)+1}^{n} w_{nj}\varepsilon''_{nj}\Big\| > \delta/2\Big\} \leq$$
$$\leq \frac{4}{\delta}\Big(\frac{p}{p-1}\Big)\Big\|b_n^{-1}(p)\sum_{j=N(n)+1}^{n} w_{nj}(\varepsilon''_{nj} - \tilde{\varepsilon}''_{nj})\Big\|_{p,\infty}$$
$$\leq \frac{4C^{1/p}}{\delta}\Big(\frac{p}{p-1}\Big)\cdot b_n^{-1}(p)\Big(\sum_{j=N(n)+1}^{n} \|w_{nj}(\varepsilon''_{nj} - \tilde{\varepsilon}''_{nj})\|_{p,\infty}^p\Big)^{1/p}$$
$$\leq \frac{16C^{1/p}}{\delta}\Big(\frac{p}{p-1}\Big)\cdot b_n^{-1}(p)\Big(\sum_{j=N(n)+1}^{n} \|w_{nj}\|^p\|\varepsilon''_{nj}\|_{p,\infty}^p\Big)^{1/p}.$$

Since

$$\Big(\sum_{j=N(n)+1}^{n} \|w_{nj}\|^p\Big)^{1/p} \leq b_n(p),$$

we have that

$$\Pr\Big\{\Big\|b_n^{-1}\sum_{j=N(n)+1}^{n} w_{nj}\varepsilon''_{nj}\Big\| > \delta/2\Big\} \leq \frac{16C^{1/p}}{\delta}\Big(\frac{p}{p-1}\Big)\Big(\sup_{j\leq n}\|\varepsilon''_{nj}\|_{p,\infty}^p\Big)^{1/p}.$$

$\sup_{j\leq n}\|\varepsilon''_{nj}\|_{p,\infty} \to 0$ as $n \to \infty$ using Lemma 4.7 since $x^p\Pr\{\|\varepsilon_0\| > x\} \to 0$ as $x \to \infty$ and $r_n \to \infty$ as $n \to \infty$. The proof is complete. □

Proof of Theorem 4.3. For $\tau > 0$, set

$$r_{nj} = \Big[\frac{\tau\delta^2(2-p)}{8\|\varepsilon_0\|_{p,\infty}^p}\Big]^{\frac{1}{2-p}}\cdot\frac{b_n(p)}{\|w_{nj}\|}.$$

Using Chebyshev's inequality, (4.17) and (4.18), we obtain

$$\Pr\{\|b_n^{-1}(p)S'_n\| > \delta/2\} \leq \frac{4}{\delta^2}b_n^{-2}(p)\sum_{j=-\infty}^{n}\|w_{nj}\|^2\operatorname{E}\|\varepsilon'_{nj}\|^2 \leq \tau.$$

Hence, the first term on the right side of (4.26) goes to 0 as $n \to \infty$ for each $\delta > 0$.

By Proposition 4.6, the second term on the right side of (4.26) goes to 0 as $n \to \infty$ for each $\delta > 0$ since

$$r_n = \left[\frac{\tau\delta^2(2-p)}{8\|\varepsilon_0\|_{p,\infty}^p}\right]^{\frac{1}{2-p}} \cdot \frac{b_n(p)}{\sup_{j\le n}\|w_{nj}\|} \to \infty \quad \text{as} \quad n \to \infty.$$

The proof is complete. □

Strong law of large numbers

Proof of Theorem 4.4. Assume without loss of generality that $\{\varepsilon_k\}$ are symmetric random elements. We use the fact that random elements $b_n^{-1}(p)S_n$ converge to 0 almost surely as $n \to \infty$ if and only if

$$\sup_{k\ge n} \|b_k^{-1}(p)S_k\| \to 0$$

in probability as $n \to \infty$ (see §10 of Chapter II of Shiryaev [54] for the proof).

Set $r_{nj} = n^{1/p}$ for each $n \ge 1$ and each $j \in \mathbb{Z}$. μ'_{n0} and μ''_{n0} given by (4.15) are both equal to 0 for each $n \ge 1$ since $\{\varepsilon_k\}$ are assumed to be symmetric.

We have that

$$\begin{aligned}\sum_{k=1}^{\infty} k^{-1}\Pr\{\|b_k^{-1}(p)S_k\| > \delta\} &\le \sum_{k=1}^{\infty} k^{-1}\Pr\{\|b_k^{-1}(p)S'_k\| > \delta/2\} \\ &\quad + \sum_{k=1}^{\infty} k^{-1}\Pr\{\|b_k^{-1}(p)S''_k\| > \delta/2\}\end{aligned} \tag{4.28}$$

for each $\delta > 0$.

Using Markov's inequality, the von Bahr-Esseen inequality and (4.4), we obtain

$$\begin{aligned}\sum_{k=N}^{\infty} k^{-1}\Pr\{\|b_k^{-1}(p)S'_k\| > \delta/2\} &\le \frac{2^{q+1}}{\delta^q}\sum_{k=N}^{\infty} k^{-1}\left(\frac{b_k(q)}{b_k(p)}\right)^q \mathrm{E}\,\|\varepsilon'_{k0}\|^q \\ &\le M\frac{2^{q+1}}{\delta^q}\sum_{k=N}^{\infty}\frac{\mathrm{E}\,\|\varepsilon'_{k0}\|^q}{k^{q/p}}\end{aligned} \tag{4.29}$$

for $N \ge 1$ such that

$$\frac{b_k(q)}{b_k(p)} \le Mk^{1/q-1/p}$$

for $k \ge N$, where M is a positive constant. We use Lemma 4.6 to show that series (4.29) converges. Hence, the first series on the right side of (4.28) converges.

Using Markov's inequality and the von Bahr-Esseen inequality, we have that

$$\sum_{k=1}^{\infty} k^{-1} \Pr\{\|b_k^{-1}(p) S_k''\| > \delta/2\} \le \frac{2^{p+1}}{\delta^p} \sum_{k=1}^{\infty} k^{-1} \operatorname{E} \|\varepsilon_{k0}''\|^p$$

and

$$\begin{aligned}
\sum_{k=1}^{\infty} k^{-1} \operatorname{E} \|\varepsilon_{k0}''\|^p &= \sum_{k=1}^{\infty} k^{-1} \sum_{l=k}^{\infty} \operatorname{E}[\|\varepsilon_0\|^p I_{\{l^{1/p} < \|\varepsilon_0\| \le (l+1)^{1/p}\}}] \\
&= \sum_{l=1}^{\infty} \sum_{k=1}^{l} k^{-1} \operatorname{E}[\|\varepsilon_0\|^p I_{\{l^{1/p} < \|\varepsilon_0\| \le (l+1)^{1/p}\}}] \\
&\le \operatorname{E} \|\varepsilon_0\|^p + \sum_{l=1}^{\infty} \log l \operatorname{E}[\|\varepsilon_0\|^p I_{\{l^{1/p} < \|\varepsilon_0\| \le (l+1)^{1/p}\}}] \\
&\le \operatorname{E} \|\varepsilon_0\|^p + p \operatorname{E}[\|\varepsilon_0\|^p \log \|\varepsilon_0\| I_{\{\|\varepsilon_0\| > 1\}}].
\end{aligned}$$

The second series on the right side of (4.28) also converges. The proof is complete.

□

5 Conclusions

The constructed example of a linear process $\{X_k\}$ with the divergent series of the operator norms of $\{a_j\}$ shows that such models might not only be interesting theoretically but also useful in the functional data analysis as a sequence of random functions with space varying memory.

The main novelty of the central limit theorem for the $L_2(\mu)$-valued linear process $\{X_k\}$ with the operators $\{a_j\}$ given by (3.2) is that the normalising sequence is not a sequence of real numbers but a sequence of multiplication operators. So when the series of the operator norms of $\{a_j\}$ diverges, we might need a different type of normalisation for the central limit theorem for the linear process with values in the separable Hilbert space $\mathbb{H}$.

The established functional central limit theorem shows that if we do not assume that the operator D commutes with the covariance operator of ε_0, we obtain a different Gaussian random process than the operator fractional Q-Brownian motion defined in Račkauskas and Suquet [48]. Thus the assumption of commutativity in Račkauskas and Suquet [48] is essential.

The normalising sequence in the functional central limit is also a sequence of multiplication operators and the limit process generates an operator self-similar process.

The established sufficient conditions for the Marcinkiewicz-Zygmund type weak and strong laws of large numbers are similar to Theorem 2.1 proven by Ibragimov and Linnik [27] in the case of the central limit theorem: we establish sufficient conditions for the Marcinkiewicz-Zygmund type weak and strong laws of large numbers for an abstract linear process with values in the separable Hilbert space $\mathbb{H}$ and the normalising sequence $\{b_n(p) : n \geq 1\}$ with $1 < p < 2$ just under the assumption of the convergece of $\sum_{j=0}^{\infty} \|a_j\|^p$.

If the series of the operator norms of $\{a_j\}$ converges, we show that the Marcinkiewicz-

Zygmund type weak and strong laws of large numbers for a linear process with values in the space $\mathbb{H}$ hold with the standard normalising sequence $\{n^{1/p} : n \geq 1\}$. If the series of operator norms of $\{a_j\}$ converges, the linear process have the same asymptotic behaviour as a sequence of independent and identically distributed random elements. However, if the series of operator norms of $\{a_j\}$ fails to converge, the normalising sequence might grow faster than $\{n^{1/p} : n \geq 1\}$ as we illustrate with an example in Section 4.1. We do not make assumptions about particular distribution of $\{\varepsilon_k\}$ so in this sense we generalise the results of Louhichi and Soulier [37].

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