

# Universality of Zeta-Functions of Cusp Forms and Non-Trivial Zeros of the Riemann Zeta-Function

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Received April 9, 2020; revised November 13, 2020; accepted November 16, 2020

**Abstract.** It is known that zeta-functions  $\zeta(s, F)$  of normalized Hecke-eigen cusp forms  $F$  are universal in the Voronin sense, i.e., their shifts  $\zeta(s + i\tau, F)$ ,  $\tau \in \mathbb{R}$ , approximate a wide class of analytic functions. In the paper, under a weak form of the Montgomery pair correlation conjecture, it is proved that the shifts  $\zeta(s + i\gamma_k h, F)$ , where  $\gamma_1 < \gamma_2 < \dots$  is a sequence of imaginary parts of non-trivial zeros of the Riemann zeta function and  $h > 0$ , also approximate a wide class of analytic functions.

**Keywords:** Montgomery pair correlation conjecture, Riemann zeta-function, zeta-function of cusp form, universality.

**AMS Subject Classification:** 11M06; 11M41.

# 1 Introduction

We start with some definitions. Let

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

be the full modular group. Suppose that  $F(z)$  is a holomorphic function in the upper half-plane, and, for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ , satisfies the functional equation

$$F\left(\frac{az + b}{cz + d}\right) = (cz + d)^\kappa F(z)$$

for some  $\kappa \in 2\mathbb{N}$ . Then  $F(z)$  has the Fourier series expansion at infinity

$$F(z) = \sum_{m=-\infty}^{\infty} c(m)e^{2\pi imz}.$$

If  $c(m) = 0$  for  $m < 0$ , then  $F(z)$  is called a modular form of weight  $\kappa$ . If the modular form  $F(s)$  has the Fourier series expansion at infinity

$$F(z) = \sum_{m=1}^{\infty} c(m)e^{2\pi imz},$$

then it is called a cusp form of weight  $\kappa$  for the full modular group.

Suppose that  $F(z)$  is a cusp form of weight  $\kappa$  for the full modular group. Then the zeta-function

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}, \quad s = \sigma + it$$

can be attached to  $F(z)$ . The latter series, in view of the estimate

$$c(m) \ll m^{\frac{\kappa-1}{2}},$$

is absolutely convergent for  $\sigma > \frac{\kappa+1}{2}$ . Moreover, it has analytic continuation to an entire function.

We additionally require that the function  $F(z)$  would be the Hecke-eigen cusp form, i.e., that  $F(z)$  would be the eigenfunction of all Hecke operators  $T_m$ ,

$$T_m f(z) = m^{\kappa-1} \sum_{\substack{a, d > 0 \\ ad = m}} \frac{1}{d^\kappa} \sum_{b \pmod{d}} F\left(\frac{az + b}{d}\right), \quad m \in \mathbb{N}.$$

Then the form  $F(z)$  can be normalized, thus, we may suppose that  $c(1) = 1$ .

In the sequel, we suppose that  $F(z)$  is a normalized Hecke-eigen cusp form of weight  $\kappa$ . In this case, the zeta-function  $\zeta(s, F)$  has, for  $\sigma > \frac{\kappa+1}{2}$ , the Euler product representation over primes

$$\zeta(s, F) = \prod_p (1 - \alpha(p)/p^s)^{-1} (1 - \beta(p)/p^s)^{-1},$$

where  $\alpha(p)$  and  $\beta(p)$  are conjugate complex numbers such that  $\alpha(p) + \beta(p) = c(p)$ .

In [8], it was proved that the function  $\zeta(s, F)$  is universal in the Voronin sense, i.e., a wide class of analytic functions is approximated by shifts  $\zeta(s + i\tau, F)$ ,  $\tau \in \mathbb{R}$ . More precisely, let  $D_\kappa = \{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\}$ . Denote by  $\mathcal{K}_F$  the class of compact subsets of the strip  $D_\kappa$  with connected complements, and by  $H_{0F}(K)$  with  $K \in \mathcal{K}_F$  the class of continuous non-vanishing functions on  $K$  that are analytic in the interior of  $K$ . Then the main result of [8] is the following statement.

**Theorem 1.** *Suppose that  $K \in \mathcal{K}_F$  and  $f(s) \in H_{0F}(K)$ . Then, for every  $\varepsilon > 0$ ,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, F) - f(s)| < \varepsilon \right\} > 0.$$

Here  $\text{meas}A$  denotes the Lebesgue measure of a measurable set  $A \subset \mathbb{R}$ .

In the latter theorem,  $\tau$  in shifts  $\zeta(s + i\tau, F)$  takes arbitrary real values, therefore, the theorem is of continuous type. Also, Theorem 1 has a discrete version when  $\tau$  in  $\zeta(s + i\tau, F)$  takes values from certain discrete sets. The classical discrete set is an arithmetical progression  $\{kh : k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ , where  $h > 0$  is a fixed number. Discrete universality theorems for the function  $\zeta(s, F)$  were considered in [9] and [10], and the following statement has been obtained.

**Theorem 2.** *Suppose that  $K \in \mathcal{K}_F$ ,  $f(s) \in H_{0F}(K)$  and  $h > 0$ . Then, for every  $\varepsilon > 0$ ,*

$$\liminf_{N \rightarrow \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K} |\zeta(s + ikh, F) - f(s)| < \varepsilon \right\} > 0.$$

Here  $\#A$  denotes the cardinality of a set  $A$ , and  $N$  runs over non-negative integers.

In [12], more general shifts  $\zeta(s + i\varphi(k), F)$  were used. Here  $\varphi(t)$  is a real-valued positive increasing function on  $[k_0 - \frac{1}{2}, \infty)$ ,  $k_0 \in \mathbb{N}$ , having a continuous derivative  $\varphi'(t)$  satisfying the estimate

$$\varphi(2t) \max_{t \leq u \leq 2t} \left( \frac{1}{\varphi'(u)} + \varphi'(u) \right) \ll t,$$

and such that the sequence  $\{a\varphi(k) : k \geq k_0\}$  with every  $a \in \mathbb{R} \setminus \{0\}$  is uniformly distributed modulo 1.

In [13], a joint version of a theorem from [12] has been proved.

The aim of this paper is an extension of Theorem 2 for the discrete set related to non-trivial zeros of the Riemann zeta-function  $\zeta(s)$  which is defined, for  $\sigma > 1$ , by the series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

and has a meromorphic continuation to the whole complex plane. The function  $\zeta(s)$  has infinitely many so-called non-trivial zeros  $\varrho = \beta + i\gamma$  lying in the strip  $\{s \in \mathbb{C} : 0 < \sigma < 1\}$ . By the Riemann hypothesis, all non-trivial zeros of  $\zeta(s)$  lie on the critical line  $\sigma = \frac{1}{2}$ .

Thus, let  $0 < \gamma_1 < \gamma_2 < \dots \leq \gamma_k \leq \dots$  be the sequence of imaginary parts of non-trivial zeros of the function  $\zeta(s)$ . We will use a hypothesis on the distribution of the sequence  $\{\gamma_k : k \in \mathbb{N}\}$ , namely, we suppose that, for  $c > 0$ ,

$$\sum_{\substack{\gamma_k \leq T \\ |\gamma_k - \gamma_l| < \frac{c}{\log T}}} \sum_{\gamma_l \leq T} \ll T \log T, \quad T \rightarrow \infty. \tag{1.1}$$

The latter estimate is implied by the famous Montgomery pair correlation conjecture [16]. The main result of the paper is the following theorem.

**Theorem 3.** *Suppose that the estimate (1.1) is true. Let  $K \in \mathcal{K}_F$ ,  $f(s) \in H_{0F}(K)$  and  $h > 0$ . Then, for every  $\varepsilon > 0$ ,*

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \#\left\{1 \leq k \leq N : \sup_{s \in K} |\zeta(s + i\gamma_k h, F) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\left\{1 \leq k \leq N : \sup_{s \in K} |\zeta(s + i\gamma_k h, F) - f(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many  $\varepsilon > 0$ .

Theorem 3 with the Riemann hypothesis in place of (1.1) was proved in [4] by using [3].

We recall that the condition (1.1) for the first-time was applied in [5] for the approximation by shifts  $\zeta(s + i\gamma_k h)$ , and in [7] for joint approximation by shifts  $(\zeta(s + i\gamma_k h), \zeta(s + i\gamma_k h, \alpha))$ , where  $\zeta(s, \alpha)$  is the Hurwitz zeta-function

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}, \quad \sigma > 1$$

with transcendental parameter  $\alpha$ . In [11], the joint approximation by shifts of Dirichlet  $L$ -functions involving the sequence  $\{\gamma_k\}$  was discussed. Finally, the paper [1] is devoted to a generalization of [7] for shifts of the periodic and periodic Hurwitz zeta-functions.

For the proof of Theorem 3, we will apply some results from [5] and [8]. On the mentioned results, we will construct a probabilistic model.

## 2 Probabilistic model

Denote by  $\mathcal{B}(\mathbb{X})$  the Borel  $\sigma$ -field of the space  $\mathbb{X}$ , and let  $H(D_F)$  be the space of analytic functions on  $D_F$  endowed with the topology of uniform convergence

on compacta. In this section, we will consider the weak convergence as  $N \rightarrow \infty$  for

$$P_{N,F}(A) \stackrel{\text{def}}{=} \frac{1}{N} \#\{1 \leq k \leq N : \zeta(s + i\gamma_k h, F) \in A\}, \quad A \in \mathcal{B}(H(D_F)).$$

To state a limit theorem for  $P_{N,F}$ , we need some notation. Denote by  $\gamma$  the unit circle on the complex plane, by  $\mathbb{P}$  the set of all prime numbers, and define the set  $\Omega = \prod_{p \in \mathbb{P}} \gamma_p$ , where  $\gamma_p = \gamma$  for all  $p \in \mathbb{P}$ . With the product topology and operation of pointwise multiplication, the infinite-dimensional torus  $\Omega$  is a compact topological Abelian group. Therefore, on  $(\Omega, \mathcal{B}(\Omega))$ , the probability Haar measure  $m_H$  exists, and we have the probability space  $(\Omega, \mathcal{B}(\Omega), m_H)$ . Denote by  $\omega = (\omega(p) : p \in \mathbb{P})$  the elements of the torus  $\Omega$ , and on the above probability space define the  $H(D_F)$ -valued random element

$$\zeta(s, \omega, F) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}.$$

We note that the latter infinite product is uniformly convergent on compact subsets of the strip  $D_F$  for almost all  $\omega \in \Omega$ , thus, it defines an  $H(D_F)$ -valued random element. Denote by  $P_{\zeta,F}$  the distribution of the random element  $\zeta(s, \omega, F)$ , i.e., for  $A \in \mathcal{B}(H(D_F))$ ,

$$P_{\zeta,F}(A) = m_H \{\omega \in \Omega : \zeta(s, \omega, F) \in A\}.$$

We will prove the following statement

**Theorem 4.** *Suppose that the estimate (1.1) is true. Then  $P_{N,F}$  converges weakly to the measure  $P_{\zeta,F}$  as  $N \rightarrow \infty$ .*

The proof of Theorem 4 consists from three limit theorems that will be stated as separate lemmas.

For  $A \in \mathcal{B}(\Omega)$ , define

$$Q_N(A) = \frac{1}{N} \#\left\{1 \leq k \leq N : (p^{-i\gamma_k h} : p \in \mathbb{P}) \in A\right\}.$$

**Lemma 1.**  *$Q_N$  converges weakly to the Haar measure  $m_H$  as  $N \rightarrow \infty$ .*

The lemma is proved in [5] by using the Fourier transform method. For this, the uniform distribution modulo 1 of the sequence  $\{a\gamma_k : k \in \mathbb{N}\}$  with every  $a \in \mathbb{R} \setminus \{0\}$  is applied.

The next lemma deals with absolutely convergent Dirichlet series. Let  $\theta > \frac{1}{2}$  be a fixed number, for  $m, n \in \mathbb{N}$ , let

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^\theta\right\}, \quad \zeta_n(s, F) = \sum_{m=1}^{\infty} \frac{c(m)v_n(m)}{m^s}.$$

Then it is known [8] that the latter series is absolutely convergent for  $\sigma > \frac{\kappa}{2}$ . Consider the mapping  $u_{n,F} : \Omega \rightarrow H(D_F)$  given by  $u_{n,F}(\omega) = \zeta_n(s, \omega, F)$ , where

$$\zeta_n(s, \omega, F) = \sum_{m=1}^{\infty} \frac{c(m)\omega(m)v_n(m)}{m^s}, \quad \omega(m) = \prod_{p^l | m, p^{l+1} \nmid m} \omega^l(p), \quad m \in \mathbb{N}.$$

Obviously, the series for  $\zeta_n(s, \omega, F)$  is also absolutely convergent for  $\sigma > \frac{\kappa}{2}$ . Therefore, the mapping  $u_{n,F}$  is continuous, hence it is  $(\mathcal{B}(\Omega), \mathcal{B}(H(D_F)))$ -measurable. Therefore, the Haar measure  $m_H$  defines the unique probability measure  $V_{n,F} = m_H u_{n,F}^{-1}$  on  $(H(D_F), \mathcal{B}(H(D_F)))$ , where, for  $A \in \mathcal{B}(H(D_F))$ ,

$$V_{n,F}(A) = m_H u_{n,F}^{-1}(A) = m_H(u_{n,F}^{-1}A).$$

For  $A \in \mathcal{B}(H(D_F))$ , set

$$P_{N,n,F}(A) = \frac{1}{N} \# \left\{ 1 \leq k \leq N : \zeta_n(s + i\gamma_k h, F) \in A \right\}.$$

**Lemma 2.**  $P_{N,n,F}$  converges weakly to the measure  $V_{n,F}$  as  $N \rightarrow \infty$ .

*Proof.* By the definitions of  $Q_N$  and  $P_{N,n,F}$ , we have

$$P_{N,n,F}(A) = \frac{1}{N} \# \left\{ 1 \leq k \leq N : (p^{-i\gamma_k h} : p \in \mathbb{P}) \in u_{n,F}^{-1}A \right\} = Q_N(u_{n,F}^{-1}A).$$

Thus,  $P_{N,n,F} = Q_N u_{n,F}^{-1}$ . Therefore, the lemma is a corollary of Lemma 1, continuity of  $u_{n,F}$  and Theorem 5.1 of [2].  $\square$

The weak convergence of the measure  $V_{n,F}$  as  $n \rightarrow \infty$  is very important for the proof of Theorem 4. The following assertion is true.

**Lemma 3.**  $V_{n,F}$  converges weakly to the measure  $P_{\zeta,F}$  as  $n \rightarrow \infty$ . Moreover, the support of  $P_{\zeta,F}$  is the set

$$S_F = \left\{ g \in H(D_F) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \right\}.$$

*Proof.* The lemma is a result of [6] and [8] because  $V_{n,F}$ , as  $n \rightarrow \infty$ , and

$$\frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \zeta(s + i\tau, F) \in A \right\}, \quad A \in \mathcal{B}(H(D_F)),$$

as  $T \rightarrow \infty$ , have the same limit measure  $P_{\zeta,F}$ .  $\square$

To prove Theorem 4, it remains to show that the limit measure of  $P_{N,F}$  as  $N \rightarrow \infty$  coincides with that of  $V_{n,F}$  as  $n \rightarrow \infty$ . For this, some mean square estimates will be applied. For convenience, we recall the Gallagher lemma which connects discrete and continuous mean squares of certain functions.

**Lemma 4.** Let  $T_0$  and  $T \geq \delta > 0$  be real numbers, and  $\mathcal{T} \neq \emptyset$  be a finite set in the interval  $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$ . Define

$$N_\delta(x) = \sum_{t \in \mathcal{T}, |t-x| < \delta} 1$$

and let  $S(t)$  be a complex-valued continuous function on  $[T_0, T_0 + T]$  having a continuous derivative on  $(T_0, T_0 + T)$ . Then

$$\sum_{t \in \mathcal{T}} N_\delta^{-1}(t) |S(t)|^2 \leq \frac{1}{\delta} \int_{T_0}^{T_0+T} |S(t)|^2 dt + \left( \int_{T_0}^{T_0+T} |S(t)|^2 dt \int_{T_0}^{T_0+T} |S'(t)|^2 dt \right)^{\frac{1}{2}}.$$

The proof of the lemma can be found in [15, Lemma 1.4].

Now, we recall a metric in the space  $H(D_F)$ . For  $g_1, g_2 \in H(D_F)$ , let

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

where  $\{K_l : l \in \mathbb{N}\}$  is a sequence of compact subsets of the strip  $D_F$  such that  $D_F = \bigcup_{l=1}^{\infty} K_l$ ,  $K_l \subset K_{l+1}$  for all  $l \in \mathbb{N}$ , and if  $K$  is a compact subset of  $D_F$ , then  $K \subset K_l$  for some  $l \in \mathbb{N}$ . Then  $\rho$  is a metric on  $H(D_F)$  that induces the topology of uniform convergence on compacta.

**Lemma 5.** *Suppose that the estimate (1.1) is true. Then the equality*

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \rho(\zeta(s + i\gamma_k h, F), \zeta_n(s + i\gamma_k h, F)) = 0$$

holds.

*Proof.* We start with some remarks on the mean squares of the function  $\zeta(s, F)$ . It is well known that, for fixed  $\sigma$ ,  $\frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}$ , the bound

$$\int_0^T |\zeta(\sigma + it, F)|^2 dt \ll_{\sigma} T$$

is true. Hence, it follows for the same  $\sigma$  that, for  $\tau \in \mathbb{R}$ ,

$$\int_0^T |\zeta(\sigma + i\tau + it, F)|^2 dt \ll_{\sigma} T(1 + |\tau|). \quad (2.1)$$

Moreover, the Cauchy integral formula together with (2.1) leads to

$$\int_0^T |\zeta'(\sigma + i\tau + it, F)|^2 dt \ll_{\sigma} T(1 + |\tau|). \quad (2.2)$$

Now, we apply Lemma 4. It is known that  $\gamma_k \sim \frac{2\pi k}{\log k}$  as  $k \rightarrow \infty$ . Therefore,  $\gamma_k \leq \frac{ck}{\log k}$  with some  $c > 0$  for all  $k \geq 2$ . In Lemma 4, we take  $\mathcal{T} = \{\gamma_1 h, \dots, \gamma_N h\}$ ,  $\delta = h \left( \log \frac{N}{c \log N} \right)^{-1}$ ,  $T_0 = \gamma_1 h - \frac{\delta}{2}$  and  $T = \gamma_N h - T_0 + \frac{\delta}{2}$ . Then, in view of (1.1), we find that

$$\sum_{k=1}^N N_{\delta}(\gamma_k h) = \sum_{k=1}^N \sum_{\substack{\gamma_l \leq \frac{cN}{h \log N} \\ |\gamma_k - \gamma_l| < \frac{\delta}{h}}} 1 = \sum_{0 < \gamma_l, \gamma_k \leq \frac{cN}{h \log N}} \sum_{|\gamma_l - \gamma_k| < \frac{\delta}{h}} 1 \ll_h N.$$

Thus, applying Lemma 4 for the function  $\zeta(\sigma + i\tau + i\gamma_k h, F)$ , and, taking into account the estimates (2.1) and (2.2), we obtain

$$\sum_{k=1}^N |\zeta(\sigma + i\tau + i\gamma_k h, F)| = \sum_{k=1}^N \left( N_{\delta}(\gamma_k h) N_{\delta}^{-1}(\gamma_k h) \right)^{\frac{1}{2}} |\zeta(\sigma + i\tau + i\gamma_k h, F)|$$

$$\begin{aligned}
 &\leq \left( \sum_{k=1}^N N_\delta(\gamma_k h) \sum_{k=1}^N N_\delta^{-1}(\gamma_k h) |\zeta(\sigma + i\tau + i\gamma_k h, F)|^2 \right)^{1/2} \\
 &\ll_h N^{\frac{1}{2}} \left( \log N \int_{\gamma_1 h - \frac{\delta}{2}}^{\gamma_N h - \gamma_1 h + \delta} |\zeta(\sigma + i\tau + it, F)|^2 dt \right. \\
 &\quad \left. + \left( \int_{\gamma_1 h - \frac{\delta}{2}}^{\gamma_N h - \gamma_1 h + \delta} |\zeta(\sigma + i\tau + it, F)|^2 dt \int_{\gamma_1 h - \frac{\delta}{2}}^{\gamma_N h - \gamma_1 h + \delta} |\zeta'(\sigma + i\tau + it, F)|^2 dt \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
 &\ll_h N^{\frac{1}{2}} \left( \log N \int_0^{\frac{c(h)N}{\log N}} |\zeta(\sigma + i\tau + it, F)|^2 dt \right. \\
 &\quad \left. + \left( \int_0^{\frac{c(h)N}{\log N}} |\zeta(\sigma + i\tau + it, F)|^2 dt \int_0^{\frac{c(h)N}{\log N}} |\zeta'(\sigma + i\tau + it, F)|^2 dt \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
 &\ll_h N^{\frac{1}{2}} \left( \log N \frac{c(h)N}{\log N} (1 + |\tau|) \right)^{\frac{1}{2}} + N^{\frac{1}{2}} \left( \frac{c(h)N}{\log N} (1 + |\tau|) \right)^{\frac{1}{2}} \\
 &\ll_h N(1 + |\tau|)^{\frac{1}{2}} \ll_h N(1 + |\tau|). \tag{2.3}
 \end{aligned}$$

Here  $c(h)$  is a certain positive constant depending of  $h$ .

Let the number  $\theta$  is the same as in the definition of  $v_n(m)$ , and

$$l_n(s) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) n^s,$$

where  $\Gamma(s)$  denotes the Euler gamma-function. Then we have [6]

$$\zeta_n(s, F) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s + z, F) l_n(z) \frac{dz}{z}.$$

Hence, taking  $\theta_1 > 0$ , we obtain

$$\zeta_n(s, F) - \zeta(s, F) = \frac{1}{2\pi i} \int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} \zeta(s + z, F) l_n(z) \frac{dz}{z}. \tag{2.4}$$

We take an arbitrary fixed compact subset  $K$  of the strip  $D_F$ , denote the points of  $K$  by  $s = \sigma + iv$ , fix  $\varepsilon > 0$  such  $\frac{\kappa}{2} + 2\varepsilon \leq \sigma \leq \frac{\kappa+1}{2} - \varepsilon$  for  $s \in K$ , and choose  $\theta_1 = \sigma - \varepsilon - \frac{\kappa}{2}$  and  $\theta = \frac{\kappa}{2} + \varepsilon$ . Then the representation (2.4) shows that, for  $s \in K$ ,

$$\zeta(s + i\gamma_k h, F) - \zeta_n(s + i\gamma_k h, F) \ll \int_{-\infty}^{\infty} |\zeta(s + i\gamma_k h - \theta_1 + i\tau, F)| \frac{|l_n(-\theta_1 + i\tau)|}{|-\theta_1 + i\tau|} d\tau.$$

Hence, after a shift  $\tau + v \rightarrow \tau$ , we have

$$\begin{aligned}
 &\zeta(s + i\gamma_k h, F) - \zeta_n(s + i\gamma_k h, F) \ll \int_{-\infty}^{\infty} \left| \zeta\left(\frac{\kappa}{2} + \varepsilon + i(\tau + \gamma_k h), F\right) \right| \\
 &\quad \times \frac{|l_n(\frac{\kappa}{2} + \varepsilon - s + i\tau)|}{|\frac{\kappa}{2} + \varepsilon - s + i\tau|} d\tau.
 \end{aligned}$$



Therefore,

$$\begin{aligned} & \frac{1}{N} \sum_{k=1}^N \sup_{s \in K} |\zeta(s + i\gamma_k h, F) - \zeta_n(s + i\gamma_k h, F)| \\ & \ll \int_{-\infty}^{\infty} \left( \frac{1}{N} \sum_{k=1}^N \left| \zeta\left(\frac{\kappa}{2} + \varepsilon + i(\tau + \gamma_k h), F\right) \right| \sup_{s \in K} \frac{|l_n(\frac{\kappa}{2} + \varepsilon - s + i\tau)|}{|\frac{\kappa}{2} + \varepsilon - s + i\tau|} \right) d\tau. \end{aligned} \quad (2.5)$$

It is well known that, uniformly in  $\sigma_1 \leq \sigma \leq \sigma_2$  with arbitrary  $\sigma_1 < \sigma_2$ ,

$$\Gamma(\sigma + i\tau) \ll \exp\{-c|\tau|\}, \quad c > 0.$$

Thus, taking into account the definition of the function  $l_n(s)$ , we find that, for  $s \in K$ ,

$$\frac{l_n(\frac{\kappa}{2} + \varepsilon - s + i\tau)}{\frac{\kappa}{2} + \varepsilon - s + i\tau} \ll n^{-\varepsilon} \exp\left\{-\frac{c|\tau - v|}{\theta}\right\} \ll_K n^{-\varepsilon} \exp\{-c|\tau|\}.$$

Therefore, by (2.5) and (2.3),

$$\begin{aligned} & \frac{1}{N} \sum_{k=1}^N \sup_{s \in K} |\zeta(s + i\gamma_k h, F) - \zeta_n(s + i\gamma_k h, F)| \\ & \ll_{K,h} n^{-\varepsilon} \int_{-\infty}^{\infty} (1 + |\tau|) \exp\{-c|\tau|\} d\tau \ll_{K,h} n^{-\varepsilon}. \end{aligned}$$

This shows that, for every compact set  $K \subset D_F$ ,

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \sup_{s \in K} |\zeta(s + i\gamma_k h, F) - \zeta_n(s + i\gamma_k h, F)| = 0,$$

and the assertion of the lemma follows from the definition of the metric  $\rho$ .  $\square$

Now, we are in position to prove Theorem 4.

*Proof of Theorem 4.* Let  $\xi_N$  be a random variable on a certain probability space  $(\hat{\Omega}, \mathcal{A}, \mu)$  with the distribution

$$\mu\{\xi_N = \gamma_k h\} = \frac{1}{N}, \quad k = 1, \dots, N.$$

Denote by  $X_{n,F}$  the  $H(D_F)$ -valued random element with the distribution  $V_{n,F}$ , where  $V_{n,F}$  is the limit measure in Lemma 2, and, on the probability space  $(\hat{\Omega}, \mathcal{A}, \mu)$ , define the  $H(D_F)$ -valued random element

$$X_{N,n,F} = X_{N,n,F}(s) = \zeta_n(s + i\xi_N, F).$$

Then, in view of Lemma 2,

$$X_{N,n,F} \xrightarrow[N \rightarrow \infty]{\mathcal{D}} X_{n,F}. \quad (2.6)$$

By Lemma 2, the measure  $V_{n,F}$  is weakly convergent to  $P_{\zeta,F}$  as  $n \rightarrow \infty$ . Thus,

$$X_{n,F} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} P_{\zeta,F}. \tag{2.7}$$

On the above probability space, define one more  $H(D_F)$ -valued random element

$$Y_{N,F} = Y_{N,F}(s) = \zeta(s + i\xi_N, F).$$

Then, applying Lemma 5, we find that, for every  $\varepsilon > 0$ ,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \mu\{\rho(Y_{N,F}, X_{N,n,F}) \geq \varepsilon\} \\ & \leq \lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N\varepsilon} \sum_{k=1}^N \rho(\zeta(s + i\gamma_k h, F), \zeta_n(s + i\gamma_k h, F)) = 0. \end{aligned}$$

This equality together with (2.6) and (2.7) shows that all hypotheses of Theorem 4.2 in [2] are satisfied. Therefore, we have

$$Y_{N,F} \xrightarrow[N \rightarrow \infty]{\mathcal{D}} P_{\zeta,F},$$

in other words,  $P_{N,F}$  converges weakly to  $P_{\zeta,F}$  as  $N \rightarrow \infty$ . The theorem is proved.

### 3 Proof of Theorem 3

The proof of Theorem 3 is quite standard, and is based on Theorem 4 and the Mergelyan theorem on the approximation of analytic functions by polynomials [14].

*Proof of Theorem 3.* By the mentioned Mergelyan theorem, there exists a polynomial  $p_\varepsilon(s)$  such that

$$\sup_{s \in K} |f(s) - e^{p_\varepsilon(s)}| < \frac{\varepsilon}{2}. \tag{3.1}$$

Define the set

$$G_\varepsilon = \left\{ g \in H(D_F) : \sup_{s \in K} |g(s) - e^{p_\varepsilon(s)}| < \frac{\varepsilon}{2} \right\}.$$

Clearly,  $e^{p_\varepsilon(s)} \in S$ . Therefore, in virtue of Lemma 3, the set  $G_\varepsilon$  is an open neighbourhood of an element of the support of the measure  $P_{\zeta,F}$ . Hence, by a property of the support,

$$P_{\zeta,F}(G_\varepsilon) > 0, \tag{3.2}$$

and Theorem 4 together with the equivalent of weak convergence of probability measures in terms of open sets [2, Theorem 2.1] implies

$$\liminf_{N \rightarrow \infty} P_{N,F}(G_\varepsilon) \geq P_{\zeta,F}(G_\varepsilon) > 0.$$

This, the definitions of  $P_{N,F}$  and  $G_\varepsilon$ , and (3.1) prove the first assertion of the theorem.

For the proof of the second assertion of the theorem, define the set

$$\hat{G}_\varepsilon = \left\{ g \in H(D_F) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then the boundary  $\partial \hat{G}_\varepsilon$  lies in the set

$$\left\{ g \in H(D_F) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \right\},$$

therefore,  $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \emptyset$  for different positive  $\varepsilon_1$  and  $\varepsilon_2$ . This remark implies that the set  $\hat{G}_\varepsilon$  is a continuity set of the measure  $P_{\zeta,F}$ , i.e.,  $P_{\zeta,F}(\partial \hat{G}_\varepsilon) = 0$ , for all but at most countably many  $\varepsilon > 0$ . Therefore, Theorem 4 together with the equivalent of weak convergence of probability measures in terms of continuity sets [2, Theorem 2.1] gives the equality

$$\lim_{N \rightarrow \infty} P_{N,F}(\hat{G}_\varepsilon) = P_{\zeta,F}(\hat{G}_\varepsilon) \quad (3.3)$$

for all but at most countably many  $\varepsilon > 0$ . The definitions of the sets  $G_\varepsilon$  and  $\hat{G}_\varepsilon$ , and inequality (3.1) imply the inclusion  $G_\varepsilon \subset \hat{G}_\varepsilon$ . Hence, in view of (3.2), we have  $P_{\zeta,F}(\hat{G}_\varepsilon) > 0$ . The latter inequality, the definitions of  $P_{N,F}$  and  $\hat{G}_\varepsilon$ , and (3.3) prove the second assertion of the theorem. The theorem is proved.

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