VILNIUS UNIVERSITY

ERIKAS KARIKOVAS

SELF-APPROXIMATION OF HURWITZ ZETA-FUNCTIONS

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Scientific supervisor - prof. dr. Ramūnas Garunkštis (Vilnius University, Physical sciences, Mathematics - 01P)

Scientific adviser - prof. habil. dr. Artūras Dubickas (Vilnius University, Physical sciences, Mathematics - 01P)

VILNIAUS UNIVERSITETAS

ERIKAS KARIKOVAS

HURVICO DZETA FUNKCIJŲ SAVIAPROKSIMACIJA

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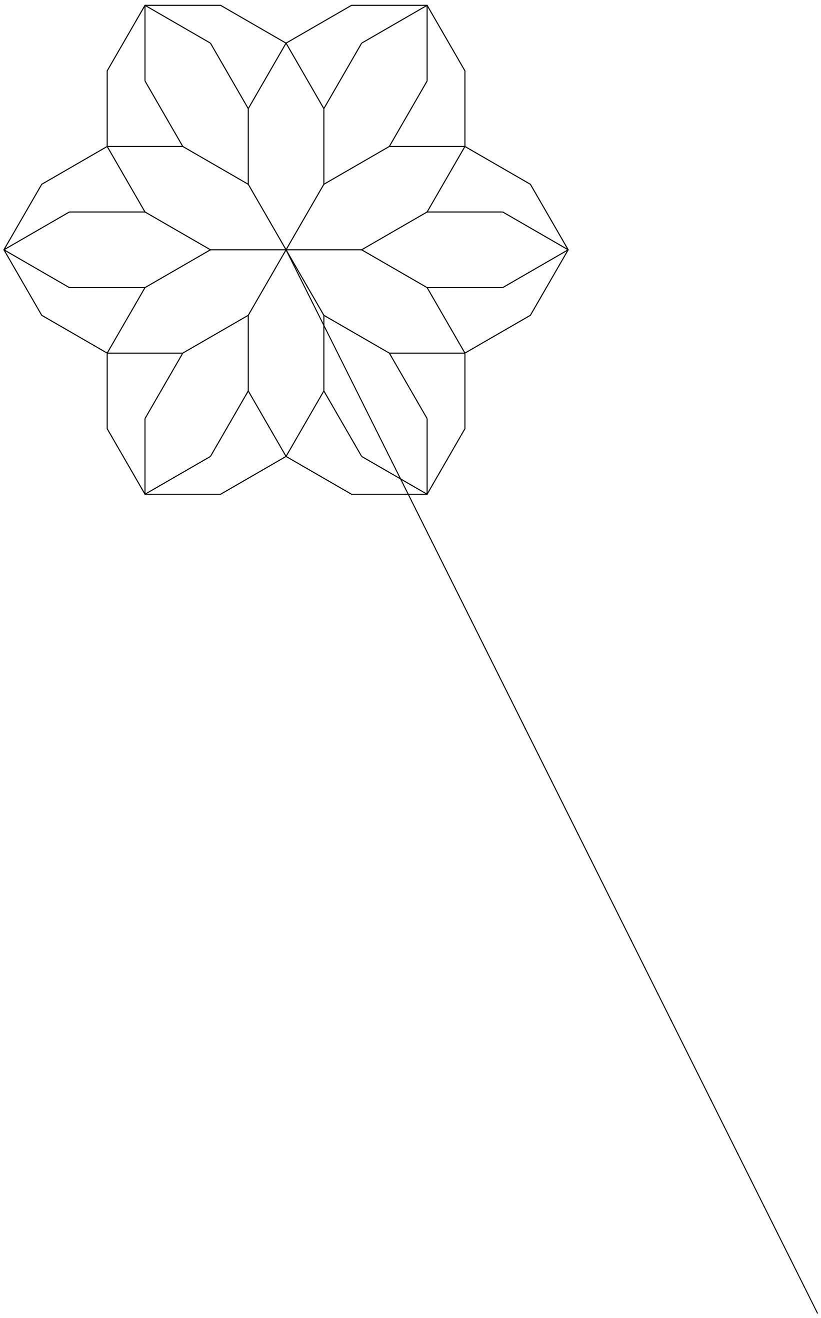
Mokslinis vadovas – prof. dr. Ramūnas Garunkštis (Vilniaus universitetas, fiziniai mokslai, matematika - 01P)

Konsultantas – prof. habil. dr. Artūras Dubickas (Vilniaus universitetas, fiziniai mokslai, matematika - 01P)

This thesis is dedicated to my wife Ramunė.



Contents

Chapter 1

Introduction

In the thesis the self-approximation of Hurwitz zeta-functions and periodic Hurwitz zeta-functions is considered.

1.1 Actuality

Zeta-functions are significant objects in analytic number theory. The central objects are the Riemann zeta function, distribution of its zeros and some issues of distribution of prime numbers.

In analytic number theory, universality theorems have significant effect on Dirichlet L-functions and zeta-functions. Almost classical applications of universality theorems are functional independence and criteria for analogues of the Riemann hypothesis.

In 1975, Voronin obtained the first most outstanding result of universality. Later more and more mathematicians started to investigate universality in relation to zeta-functions. New results in universality theory were obtained by B. Bagchi, R. Garunkštis, S.M. Gonek, J. Kaczorowski, A. Laurinčikas, K. Matsumoto, A. Reich, J. Steuding and other Japanese, Polish, Lithuanian and German mathematicians.

Approximation is very important in mathematics. Universality theorems play the crucial role in approximation of analytic functions.

This thesis will deal with the property of self-approximation related to Hurwitz and periodic Hurwitz zeta-functions. In the proofs of new theorems analytic methods will be used.

1.2 Aims and results

The main aim of this thesis is to prove the self-approximation property for Hurwitz zeta-functions and periodic Hurwitz zeta-functions. In this section we present summary of problems investigated in this thesis.

Let $s = \sigma + it$ denote a complex variable. For $\sigma > 1$, the Hurwitz zeta-function is given by

$$\zeta(s, \omega) = \sum_{n=0}^{\infty} \frac{1}{(n+\omega)^s},$$

where ω is a parameter from the interval (0,1].

Denote by $\mathfrak{A} = \{c_m : m \in \mathbb{N}_0\}, \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ a periodic sequence of complex numbers with the smallest period $k \in \mathbb{N}$.
For $\sigma > 1$, the periodic Hurwitz zeta-function is defined by

$$\zeta(s, \omega; \mathfrak{A}) = \sum_{m=0}^{\infty} \frac{c_m}{(m+\omega)^s}.$$

1. In Chapter 3, we will study the self-approximation of Hurwitz zeta-functions with a transcendental parameter. We will prove the following theorem.

   Theorem 3.4.1. *Let $l \leq m$ be positive integers and let ω be a transcendental number from the interval (0,1]. Let $d_1, \ldots, d_l \in \mathbb{R}$ be such that the set*

   $$A(d_1, d_2, \ldots, d_l; \omega) = \{d_j \log(n+\omega) : j = 1, \ldots, l; n \in \mathbb{N}_0\}$$

   is linearly independent over $\mathbb{Q}$. For $m > l$, let $d_{l+1}, \ldots, d_m \in \mathbb{R}$ be such that each d_k, $k = l+1, \ldots, m$ is a linear combination of $d_1, \ldots, d_l$ over $\mathbb{Q}$.

   Then, for any $\varepsilon > 0$,

   $$\liminf_{T\to\infty} \frac{1}{T} meas \Big\{\tau \in [0, T] : \max_{1\leq j,k\leq m} \max_{s\in\mathcal{K}} |\zeta(s + id_j\tau, \omega) - \zeta(s + id_k\tau, \omega)| < \varepsilon \Big\} > 0.$$

2. In Chapter 4, we will consider the self-approximation of Hurwitz zeta-functions with a rational parameter. We will prove the following theorem.

   Theorem 4.1.1. *Let $\omega = \frac{a}{b}$ be a rational number satisfying $0 < a < b$ and $gcd(a, b) = 1$. Moreover, suppose that α, β are real numbers linearly independent over $\mathbb{Q}$ and $\mathcal{K}$ is any compact subset of the strip $1/2 < \sigma < 1$.*

Then, for any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0,T] : \max_{s\in\mathcal{K}} \left| \zeta\left(s+i\alpha\tau, \frac{a}{b}\right) - \zeta\left(s+i\beta\tau, \frac{a}{b}\right) \right| < \varepsilon \right\} > 0.$$

3. Chapter 5 deals with the self-approximation of periodic Hurwitz zeta-functions with transcendental and rational parameters. We will prove the following theorems.

   Theorem 5.1.1. *Let* $\mathfrak{A} = \{c_m : m \in \mathbb{N}_0\}$ *be a periodic sequence of complex numbers with the smallest period* $k \in \mathbb{N}$. *Let* $\omega = \frac{a}{b}$, $\omega \in (0,1]$, $0 < a < b$, $gcd(a,b) = 1$ *be a rational number. Moreover, suppose that* α, β *are real numbers linearly independent over* $\mathbb{Q}$ *and* $\mathcal{K}$ *is any compact subset of the strip* $1/2 < \sigma < 1$. *Then, for any* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} meas \Big\{ \tau \in [0,T] : \\ \max_{s\in\mathcal{K}} \left| \zeta\left(s+i\alpha\tau, \frac{a}{b}; \mathfrak{A}\right) - \zeta\left(s+i\beta\tau, \frac{a}{b}; \mathfrak{A}\right) \right| < \varepsilon \Big\} > 0.$$

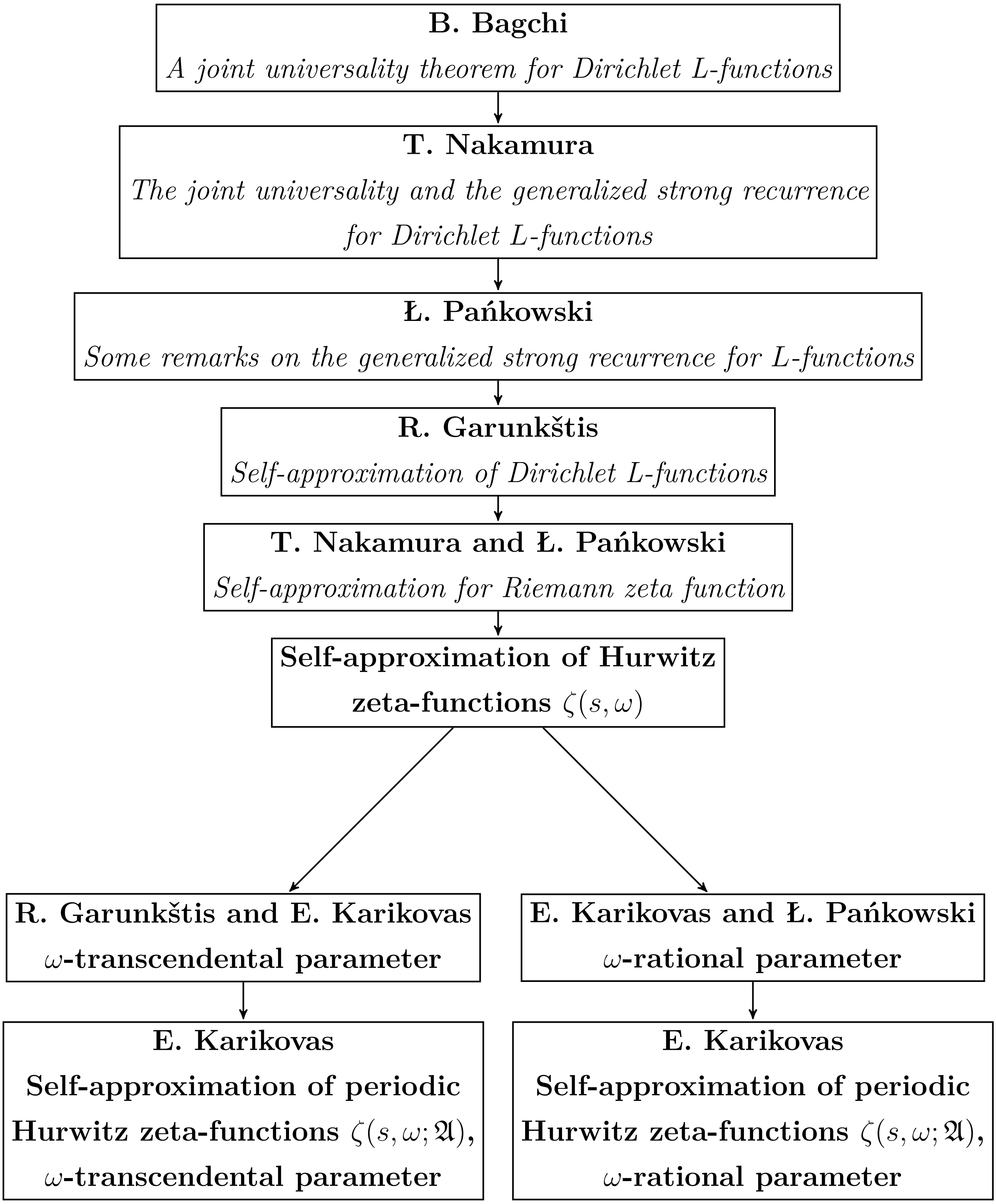
   The next theorem deals with the case of transcendental parameter.

   Theorem 5.1.2. *Let* $\mathfrak{A} = \{c_m : m \in \mathbb{N}_0\}$ *be a periodic sequence of complex numbers with the smallest period* $k \in \mathbb{N}$. *Let* ω *be a transcendental number from the interval* $(0,1]$. *Moreover, suppose that* $\alpha, \beta \in \mathbb{R}$ *are such that* $A(\alpha,\beta;\omega)$ *is linearly independent over* $\mathbb{Q}$ *and* $\mathcal{K}$ *is any compact subset of the strip* $1/2 < \sigma < 1$. *Then, for any* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} meas \Big\{ \tau \in [0,T] : \max_{s\in\mathcal{K}} \left| \zeta\left(s+i\alpha\tau, \omega; \mathfrak{A}\right) - \zeta\left(s+i\beta\tau, \omega; \mathfrak{A}\right) \right| < \varepsilon, \\ \left|\left| \frac{(\alpha-\beta)\tau\log k}{2\pi} \right|\right| < \varepsilon \Big\} > 0.$$

Here $\|x\|$ denotes the distance from $x \in \mathbb{R}$ to the nearest integer.

We state the schematic diagram of the evolution of the results, concerning the generalized strong recurrence (Self-approximation).



1.3 Methods

In the thesis we used recent methods introduced by Garunkštis [12] and Pańkowski [43], [44]. Also, elements of complex analysis, measure theory and diophantine methods are used.

1.4 Novelty and originality

All results obtained in this thesis are new and original. The results of this thesis contribute to the theory of self-approximation and to the theory of Hurwitz zeta-functions.

1.5 Publications

The results of this thesis are published in three papers.

1. R. Garunkštis, E. Karikovas, *Self-approximation of Hurwitz zeta-functions*, Funct. Approx. Comment. Math., **51(1)** (2014), 181–188.

2. E. Karikovas, Ł. Pańkowski, *Self-approximation of Hurwitz zeta-functions with rational parameter*, Lith. Math. J., **54(1)** (2014), 74–81.

3. E. Karikovas, *Self-approximation of periodic Hurwitz zeta-functions*, to appear in Nonlinear Anal. Model. Control.

1.6 Conferences and visits

1. E. Karikovas, *Self-approximation of Hurwitz zeta-functions*, Summer school, Four faces of number theory, August 7–11 , 2012, Department of Mathematics Julius-Maximilians-Universität Würzburg, Germany.

2. E. Karikovas, *Self-approximation of Hurwitz zeta-functions with rational parameter*, 54^{th} Conference of Lithuanian Mathematical Society, June 19–20, 2013, Vilnius, Lithuania.

3. E. Karikovas, *Self-approximation of Hurwitz zeta-functions*, 28th Journées Arithmétiques, July 1–5, 2013, Grenoble, France.

4. E. Karikovas, *Self-approximation of periodic Hurwitz zeta-functions*, Elementare und Analytische Zahlentheorie, ELAZ Conference at University of Hildesheim, July 28–August 1, 2014, Hildesheim, Germany.

The results of the thesis were presented at the seminars on Number Theory of the Department of Probability Theory and Number Theory at the Faculty of Mathematics and Informatics of Vilnius University.

1.7 Acknowledgment

First of all I extend my sincere gratitude to my supervisor Ramūnas Garunkštis who proposed very interesting ideas and problems as well as provided with helpful advice and guidance. His friendly observations and professional assistance were felt during the whole period of doctoral studies. I also appreciate his encouragement to participate in conferences abroad and to cooperate with foreign mathematicians. Thanks to him, I met with Polish mathematician Ł. Pańkowski, his good friend German mathematician J. Steuding, a Japanese mathematician K. Matsumoto and others. I am also thankful for the opportunity to supervise practical work of his students, which helped me to further develop my teaching skills.

For kind cooperation, I am very grateful to Ł. Pańkowski, the coauthor of the second article, the results of which are included in this thesis. His dedicated and thorough work was an invaluable experience for me.

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I also want to thank my family members, especially my wife Ramunė, who kept supporting me in difficult moments and took care of me when I was ill. I also appreciate her accompanying me to conferences.

I am thankful to everybody who contributed and supported me in various ways during doctoral studies.

Finally, I thank God for the possibility to study mathematics, for sending me all the above mentioned people and the possibility to communicate with them.

Chapter 2

Literature review

2.1 History of the problem

In this section, we present universality theorems for the Riemann zeta-function $\zeta(s)$ and Dirichlet L-function $L(s,\chi)$ and other interesting facts, which lead to results obtained in the thesis.

Let, as usual, $s=\sigma+it$ denote a complex variable. For $\sigma>1$, the Riemann zeta-function is defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p\left(1-\frac{1}{p^s}\right)^{-1},$$

where p is a prime number. The function $\zeta(s)$ can be analytically continued to the whole complex plane, except the point $s=1$ (simple pole) with residue 1.

This function is a very famous and important object in analytic number theory. It is known that $\zeta(s)=0$ if $s=-2n$, for $n=1,2,\dots$. These zeros are called trivial-zeros.

The well-known yet unproved Riemann [49] hypothesis states:

Riemann's hypothesis. *All non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma=\frac{1}{2}$.*

Hilbert about this hypothesis declared (see [7]):

"*If I were to awaken after having slept for a thousand years, my first question would be: has the Riemann hypothesis been proven?*"

It is known that $\zeta(s)$ has no zeros in the region $\Re(s)\geq 1$. There are known special values, for example $\zeta(2)=\frac{\pi^2}{6}$, $\zeta(4)=\frac{\pi^4}{90}$. More about the Riemann zeta-function can be found in [1], [9], [17], [23], [52], [54].

In 1975, Voronin [55] discovered the universality theorem of the Riemann zeta-function. In other words, this means that any analytic nonzero function in the critical strip $\mathcal{D}=\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\}$ can be approximated by shifts $\zeta(s+i\tau)$.

The precise statement of the Voronin's theorem is the following.

Theorem 2.1.1 (Voronin [55])**.** *Let $0 < r < \frac{1}{4}$. Suppose that $f(s)$ is a continuous non-vanishing function on the disc $|s| \le r$, and analytic in the interior of this disc. Then, for any $\varepsilon > 0$, there exists a number $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that*

$$\max_{s \le r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon. \tag{2.1}$$

Let meas$\{A\}$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Recall that, for Lebesgue measurable set $A \subset (0, \infty)$, we define *lower density* of A as

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}(A \cap (0, T]).$$

Moreover, if this limit is positive, then we say that A has *a positive lower density*. For measure theory see more in [6], [53] (Chapter 10) and in [50] (Chapter 11).

Now we present the current version of the Voronin theorem. The proof of this theorem can be found in [29].

Theorem 2.1.2 (Voronin's universality theorem)**.** *Let $\mathcal{K}$ be a compact subset of the strip $\mathcal{D}$ with connected complement, and $f(s)$ be a continuous non-vanishing function on $\mathcal{K}$ which is analytic in the interior of $\mathcal{K}$. Then, for any $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} meas \left\{ \tau \in [0, T] : \max_{s \in \mathcal{K}} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0. \tag{2.2}$$

This theorem shows that the set of translations of the Riemann zeta function which approximate given analytic function $f(s)$ is sufficiently rich: it has a positive lower density. However we do not know explicitly τ with the approximation property.

Next we recall the definition of a Dirichlet character.

A Dirichlet character modulo $q > 0$, denoted by $\chi(q)$ is any function $\chi : \mathbb{Z} \longrightarrow \mathbb{C}$ with the properties :

- *χ is periodic modulo q, i.e., $\chi(n + q) = \chi(n)$ for all $n \in \mathbb{Z}$.*
- *If q and n are not relatively prime, then $\chi(n) = 0$.*
- *If q and n are relatively prime, then $|\chi(n)| = 1$.*
- *If m and n are any two positive integers, then $\chi(mn) = \chi(m)\chi(n)$.*

We present several basic facts of Dirichlet characters. Let $G(q)$ be the set of characters modulo q. We define the product $\chi_1\chi_2$ of $\chi_1, \chi_2 \in G(q)$ by

$$(\chi_1\chi_2)(n) = \chi_1(n)\chi_2(n), \quad \text{for } n \in \mathbb{Z}.$$

With this operation, $G(q)$ becomes a group, with unit element the *principal* character modulo q given by

$$\chi_0{}^{(q)}(n) = \begin{cases} 1, & \text{if } (n,q) = 1 \\ 0, & \text{if } (n,q) > 1. \end{cases}$$

The inverse of $\chi \in G(q)$ is its complex conjugate $\overline{\chi} : n \to \overline{\chi(n)}$.

The values of Dirichlet character χ modulo q are either 0, or $\varphi(q)$th roots of unity; i.e., for all n we have either $\chi(n) = 0$ or $\chi(n) = e^{2\pi m/\varphi(q)}$, where $m = m(n) \in \mathbb{N}$ and $\varphi(q)$ is Euler function. There exist exactly $\varphi(q)$ Dirichlet characters modulo q. Moreover, for any integer a with $(a,q) = 1$ and $a \not\equiv 1 \bmod q$ there exists a character χ with $\chi(a) \neq 1$.

Next we define induced, primitive and equivalent Dirichlet characters. These definitions will be useful in Chapter 4 and Chapter 5.

Let χ be a character $\bmod q$ and $d > 0$ be divisor of q. We say that q is *induced* by character $\chi' mod\ d$ if $\chi(n) = \chi'(n)$ for any $n \in \mathbb{Z}$ with $(n,q) = 1$. Similarly stated, χ is *induced* by χ' if $\chi = \chi' \chi_0{}^{(q)}$. Notice that if $(a,d) = 1$ and $(a,q) > 1$, then $\chi'(n) \neq 0$, but $\chi(n) = 0$.

The character χ is called *primitive* if it is not induced by a character mod d for any divisor $d < q$ of q.

Two Dirichlet characters χ_1 and χ_2 are *equivalent* if they are induced by the same primitive character.

Let us present several examples. Denote by $\chi_k(n;q)$ the kth character mod q. In the Table 1 below we state all non zero values of all Dirichlet characters mod 8.

n	1	3	5	7
$\chi_1(n;8)$	1	1	1	1
$\chi_2(n;8)$	1	-1	-1	1
$\chi_3(n;8)$	1	-1	1	-1
$\chi_4(n;8)$	1	1	-1	-1

Table 1.

Table 2 below shows all non zero values of all Dirichlet characters mod 4.

n	1	3
$\chi_1(n;4)$	1	1
$\chi_2(n;4)$	1	-1

Table 2.

Adding two columns to Table 2 by periodicity we obtain the following:

n	1	3	5	7
$\chi_1(n;4)$	1	1	1	1
$\chi_2(n;4)$	1	-1	1	-1

Table 3.

We see that $\chi_1(n;8) = \chi_1(n;4)$ for all $n = 1, 3, 5, 7$, so $\chi_1(n;8)$ is imprimitive character; in the other words, χ_1 mod 8 is induced by χ_1 mod 4.
$\chi_2(5;8) \neq \chi_2(5;4)$ so χ_2 mod 8 is primitive character.
$\chi_3(n;8) = \chi_2(n;4)$ for all $n = 1, 3, 5, 7$, so $\chi_3(n;8)$ is imprimitive character.
$\chi_4(5;8) \neq \chi_2(5;4)$ so χ_4 mod 8 is primitive character.

In the following table we state all non zero values of all Dirichlet characters mod 10.

n	1	3	7	9	primitive
$\chi_1(n;10)$	1	1	1	1	No
$\chi_2(n;10)$	1	i	-i	-1	No
$\chi_3(n;10)$	1	-i	i	-1	No
$\chi_4(n;10)$	1	-1	-1	1	No

Table 4.

For $\sigma > 1$, the Dirichlet L-function is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

It is easy to see that for $q = 1$ we get $L(s,\chi) = \zeta(s)$.

Denote by χ_0 the principal character modulo q. The function $L(s,\chi_0)$ is analytically continued to the whole complex plane, except for a simple pole at $s = 1$. If $\chi \neq \chi_0$, then $L(s,\chi)$ is analytically continued to an entire function. Furthermore, just as $\zeta(s)$, the function $L(s,\chi)$ has infinitely many zeros in the strip $0 < \sigma < 1$.

More about Dirichlet characters and Dirichlet L-functions can be found in [1], [9], [17], [39], [52].

As a generalization of Theorem 2.1.2, S.M. Voronin also proved the joint universality theorem. This theorem implies that a collection of Dirichlet L-functions with non-equivalent characters uniformly approximates simultaneously non-vanishing analytic functions; in slightly different form this was also established by Gonek [16] and Bagchi [2] (independently; all these sources are unpublished doctoral theses).

Next we state the strongest version of the *joint universality* theorem.

Theorem 2.1.3 (Voronin's joint universality theorem)**.** *Let* $\chi_1 \bmod q_1, \dots, \chi_m \bmod q_m$ *be pairwise non-equivalent Dirichlet characters,* $\mathcal{K}_1, \dots, \mathcal{K}_m$ *be compact subsets of the strip* $1/2 < \sigma < 1$ *with connected complements. Further, for each* $1 \leq l \leq m$ *let* $f_l(s)$ *be a non-vanishing continuous function on* $\mathcal{K}_l$ *which is analytic in the interior of* $\mathcal{K}_l$*. Then, for any* $\varepsilon > 0$*, we have*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0, T] : \max_{1\leq l\leq m} \max_{s\in\mathcal{K}_l} |L(s + i\tau, \chi_l) - f_l(s)| < \varepsilon \right\} > 0. \tag{2.3}$$

This is Theorem 1.10 in [52].

We state the following generalization of the Riemann hypothesis.

Generalized Riemann hypothesis. *Let* χ *be a Dirichlet character. All zeros of* $L(s, \chi)$ *with* $0 < \sigma < 1$ *lie on the critical line* $\sigma = \frac{1}{2}$.

In 1982, Bagchi [3] discovered an interesting equivalent to the generalized Riemann hypothesis. He proved that the generalized Riemann hypothesis is true if and only if the Dirichlet L-functions can be approximated by itself.

Theorem 2.1.4 (Bagchi [3])**.** *The generalized Riemann hypothesis is true if and only if, for any compact subset* $\mathcal{K}$ *of the strip* $1/2 < \sigma < 1$ *and any* $\varepsilon > 0$*,*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0, T] : \max_{s\in\mathcal{K}} |L(s + i\tau, \chi) - L(s, \chi)| < \varepsilon \right\} > 0. \tag{2.4}$$

This property is called the *strong recurrence* (see also Theorem 8.3 in [52]).

Kaczorowski, Laurinčikas and Steuding [27] discovered another property similar to strong recurrence (see also Section 10.6 in [52]).

Theorem 2.1.5 (Kaczorowski, Laurinčikas and Steuding [27])**.** *Let* $\mathcal{K}$ *be a compact subset of the strip* $1/2 < \sigma < 1$ *with connected complement and let* $\lambda \in \mathbb{R}$ *be* *such that $\mathcal{K}$ and $\mathcal{K}+i\lambda := \{s+i\lambda : s\in\mathcal{K}\}$ are disjoint. Then, for any $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}meas\Big\{\tau\in[0,T]: \max_{s\in\mathcal{K}}\left|L\left(s+i\lambda+i\tau,\chi\right)-L\left(s+i\tau,\chi\right)\right|<\varepsilon\Big\}>0. \tag{2.5}$$

We recall that numbers $v_1, v_2, \ldots, v_n \in \mathbb{R}$ are called linearly independent over the field of rational numbers $\mathbb{Q}$ if

$$\sum_{j=1}^{n} a_j v_j = 0$$

with rational $a_1, a_2, \ldots, a_n$ implies that $a_1 = a_2 = \cdots = a_n = 0$.

Nakamura in [40] considered the joint universality of shifted Dirichlet L-functions. Assume that $1 = d_1, d_2, \ldots, d_m$ are algebraic real numbers linearly independent over $\mathbb{Q}$ and χ is an arbitrary Dirichlet character. Then, for every $\varepsilon > 0$, we have

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\Big\{\tau\in[0,T]: \max_{1\le j,k\le m}\max_{s\in\mathcal{K}}|L(s+id_j\tau,\chi)-L(s+id_k\tau,\chi)|<\varepsilon\Big\}>0. \tag{2.6}$$

For $m = 2$, Pańkowski [43] using Six Exponentials Theorem (the proof can be found in [28] and [47]) showed that (2.6) also holds for every real numbers d_1, d_2 linearly independent over $\mathbb{Q}$. Now we present Pańkowski result.

Theorem 2.1.6 (Pańkowski [43]). *Let $\mathcal{K}\subset\mathcal{D}$ be any compact set with connected complement, χ a Dirichlet character and f, g be any functions which are non-vanishing and continuous on $\mathcal{K}$ and analytic in the interior. Moreover, let α, β be real numbers linearly independent over $\mathbb{Q}$. Then, for every $\varepsilon > 0$, the set of real numbers τ, satisfying the following inequalities :*

$$\max_{s\in K}|L(s+i\alpha\tau,\chi)-f(s)|<\varepsilon,$$

$$\max_{s\in K}|L(s+i\beta\tau,\chi)-g(s)|<\varepsilon,$$

has a positive lower density.

The case where $d_1/d_2 \in \mathbb{Q}$ in inequality (2.6) was considered by Garunkštis (see [12]) and Nakamura (see [41]) independently. It is worth mentioning that the proofs of their results contain gaps. The gaps were filled by Nakamura and Pańkowski in [42], where $d_1 = 1$ and $d_2 = a/b \in \mathbb{Q}$ satisfies $\gcd(a,b) = 1$, $|a-b| \neq 1$.

Theorem 2.1.7 (Pańkowski and Nakamura [42]). *For every $0 \neq d = a/b \in \mathbb{Q}$, with $|a-b| \neq 1$ and $gcd(a,b) = 1$, every compact subset $\mathcal{K}$ of the strip $1/2 < \sigma < 1$ and every $\varepsilon > 0$, we have*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0,T] : \max_{s\in\mathcal{K}} |\zeta(s+i\tau) - \zeta(s+id\tau)| < \varepsilon \right\} > 0. \tag{2.7}$$

It should be mentioned that the general case for $d_1 = 1$ and for non-zero rational d_2 in inequality (2.6) is still open.

2.2 Basic facts of Hurwitz zeta-functions

In this section we present several properties of Hurwitz zeta functions, which will be useful in this thesis. For $\sigma > 1$, the Hurwitz zeta-function is given by

$$\zeta(s,\omega) = \sum_{n=0}^{\infty} \frac{1}{(n+\omega)^s},$$

where ω is a parameter from the interval (0,1]. It is well-known that $\zeta(s,1) = \zeta(s)$ and $\zeta(s,1/2) = \big(2^s - 1\big)\zeta(s)$, where $\zeta(s)$ is the Riemann zeta-function.

The series for $\zeta(s,\omega)$ converges absolutely for $\sigma > 1$. The convergence is uniform in every half-plane $\sigma \geq 1+\delta$, $\delta > 0$, so $\zeta(s,\omega)$ is the analytic function of s in the half-plane $\sigma > 1$. The Hurwitz zeta-function can be continued analytically to the entire complex plane, except for a simple pole at $s = 1$.

For $\sigma > 1$, the Hurwitz zeta-function has the integral representation (see Theorem 12.2 in [1])

$$\zeta(s,\omega) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}e^{-\omega x}}{1-e^{-x}} dx,$$

where $\Gamma(s)$ is the gamma function defined by

$$\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-x} dx, \text{ for } \sigma > 0.$$

To extend the Hurwitz zeta-function $\zeta(s,\omega)$ beyond the line $\sigma = 1$, we define the following integral representation.

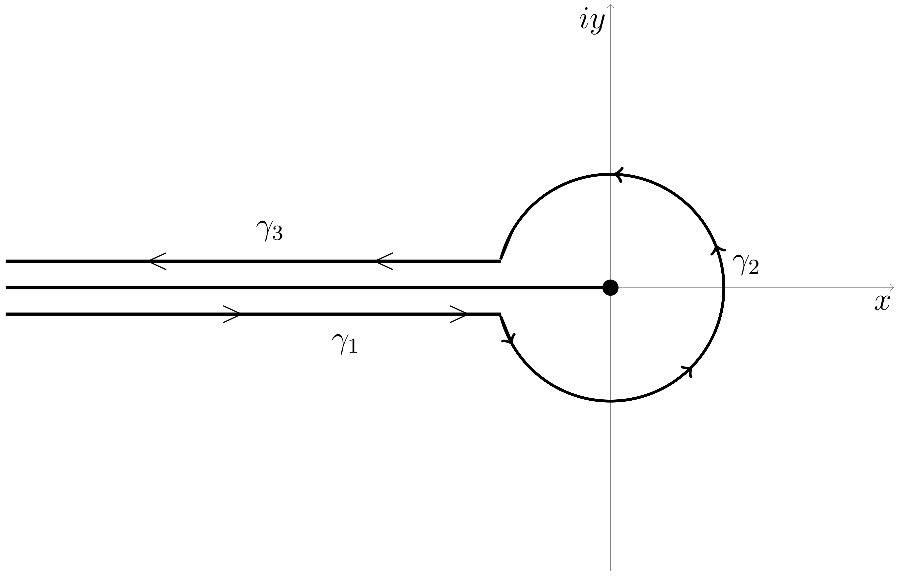

Figure 1.

The contour γ is composed of three parts $\gamma_1, \gamma_2, \gamma_3$ as shown in Figure 1. Part γ_2 is a positively oriented circle of radius $r < 2\pi$ about the origin, and parts γ_1, γ_3 are the lower and upper edges of the "cut" in the s-plane along the negative real axis, traversed as shown in the figure. This means that we use the parametrization $s = Re^{-\pi i}$ on γ_1 and $s = Re^{\pi i}$ on γ_3 where $r < R < +\infty$.

Lemma 2.2.1 (Apostol [1])**.** *If $0 < \omega \leq 1$, then the function defined by the contour integral*

$$I(s, \omega) = \frac{1}{2\pi i} \int_\gamma \frac{z^{s-1} e^{\omega z}}{1 - e^z} dz$$

is an entire function of s. Moreover, we have

$$\zeta(s, \omega) = \Gamma(1 - s) I(s, \omega), \quad \textit{for } s \neq 1.$$

Proof. This is Theorem 12.3 in [1]. □

For rational $\omega = \frac{a}{b}$ satisfying $0 < a < b$ and $gcd(a, b) = 1$ the Hurwitz zeta function might be expressed as a linear combination of Dirichlet L-functions:

$$\zeta\left(s, \frac{a}{b}\right) = \frac{b^s}{\varphi(b)} \sum_{\chi \bmod b} \overline{\chi(a)} L(s, \chi).$$

We can also express $L(s, \chi)$ in terms of Hurwitz zeta-functions. If χ is a Dirichlet character mod b, we rearrange the terms in the series for $L(s, \chi)$ according to the residue classes mod b. That is, we write $n = qb + a$, where $1 \leq a \leq b$ and $q = 0, 1, 2, \ldots$, and obtain

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\sum_{a=1}^{b}\sum_{q=0}^{\infty}\frac{\chi(qb+a)}{(qb+a)^s}=\frac{1}{b^s}\sum_{a=1}^{b}\chi(a)\sum_{q=0}^{\infty}\frac{1}{(q+\frac{a}{b})^s}$$
$$=\frac{1}{b^s}\sum_{a=1}^{b}\chi(a)\zeta\left(s,\frac{a}{b}\right).$$

In the next lemma, we state the functional equation for Hurwitz zeta-functions.

Lemma 2.2.2 (Apostol [1]). *If h and k are integers, $1\le h\le k$, then for all s we have*

$$\zeta\left(1-s,\frac{h}{k}\right)=\frac{2\Gamma(s)}{(2\pi k)^s}\sum_{r=1}^{k}\cos\left(\frac{\pi s}{2}-\frac{2\pi rh}{k}\right)\zeta\left(s,\frac{r}{k}\right).$$

Proof. This is Theorem 12.8 in [1]. □

If $h=k=1$, from the last equality we obtain the functional equation for the Riemann zeta-function.

$$\zeta(1-s)=2(2\pi)^{-s}\Gamma(s)\cos\Big(\frac{\pi s}{2}\Big)\zeta(s)$$

or equivalently

$$\zeta(s)=2(2\pi)^{s-1}\Gamma(1-s)\sin\Big(\frac{\pi s}{2}\Big)\zeta(1-s).$$

If n is nonnegative integer, then the value of $\zeta(-n,\omega)$ can be calculated explicitly. Taking $s=-n$ in the relation $\zeta(s,\omega)=\Gamma(1-s)I(s,\omega)$ we have

$$\zeta(-n,\omega)=\Gamma(1+n)I(-n,\omega)=n!I(-n,\omega)=n!Res_{z=0}\Big(\frac{z^{-n-1}e^{\omega z}}{1-e^z}\Big).$$

The next lemma gives an approximation of Hurwtz zeta-functions $\zeta(s,\omega)$ by a finite sum.

Lemma 2.2.3 (Apostol [1]). *For any integer $N\ge 0$ and $\sigma>0$ we have*

$$\zeta(s,\omega)=\sum_{n=0}^{N}\frac{1}{(n+\omega)^s}+\frac{(N+\omega)^{1-s}}{s-1}-s\int_N^{\infty}\frac{x-[x]}{(x+\omega)^{s+1}}dx.$$

The proof of this lemma can be found in [1], see Theorem 12.23.

The distribution of zeros of $\zeta(s,\omega)$ as a function of s depends drastically on the parameter ω. For instance, the Hurwitz-zeta function $\zeta(s,1/2)=(2^s-1)\zeta(s)$ vanishes for $s=2\pi ik/\log 2, k\in\mathbb{Z}$, and all other non-real zeros are expected to lie on the critical line $\sigma=1/2$.

It is known that for any $1/2<\sigma_1<\sigma_2<1$ and any transcendental or rational number $\omega\neq 1/2,1$ the function $\zeta(s,\omega)$ has more than cT zeros in the rectangle

$\sigma_1 \le \sigma \le \sigma_2$, $|t| \le T$, where c is a positive constant depending on σ_1, σ_2 and ω (see [16], [23], [15]).

More about Hurwitz zeta-functions see in [1], [23], [52].

2.3 Basic facts of periodic Hurwitz zeta-functions

In this section we present useful results about periodic Hurwitz zeta-functions. Denote by $\mathfrak{A} = \{c_m : m \in \mathbb{N}_0\}, \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ a periodic sequence of complex numbers with the smallest period $k \in \mathbb{N}$.
For $\sigma > 1$, the periodic Hurwitz zeta-function is defined by

$$\zeta(s, \omega; \mathfrak{A}) = \sum_{m=0}^{\infty} \frac{c_m}{(m+\omega)^s}.$$

This function was introduced by Laurinčikas and Javtokas in [22].

If $\mathfrak{A} = \{1\}$, then $\zeta(s, \omega; \mathfrak{A})$ is the classical Hurwitz zeta-function. In the case when $\mathfrak{A} = \{1\}$ and $\omega = 1$, the function $\zeta(s, \omega; \mathfrak{A})$ becomes the Riemann zeta-function. If $\omega = 1$, then the function $\zeta(s, \omega; \mathfrak{A})$ reduces to the periodic zeta-function

$$\zeta(s; \mathfrak{A}) = \sum_{m=1}^{\infty} \frac{c_{m-1}}{m^s}, \quad \sigma > 1.$$

It is not difficult to see that, for $\sigma > 1$,

$$\begin{aligned} \zeta(s, \omega; \mathfrak{A}) &= \sum_{l=0}^{k-1} \sum_{m=0}^{\infty} \frac{c_l}{(mk+l+\omega)^s} = \frac{1}{k^s} \sum_{l=0}^{k-1} c_l \sum_{m=0}^{\infty} \frac{1}{(m+(l+\omega/k))^s} \\ &= \frac{1}{k^s} \sum_{l=0}^{k-1} c_l \zeta\left(s, \frac{l+\omega}{k}\right). \end{aligned} \tag{2.8}$$

Therefore equation (2.8) gives the analytic continuation for $\zeta(s, \omega; \mathfrak{A})$ to the whole complex plane, except, perhaps, for a simple pole $s = 1$ with residue

$$c = \frac{1}{k} \sum_{l=0}^{k-1} c_l.$$

If $c = 0$, then $\zeta(s, \omega; \mathfrak{A})$ is an entire function.

For rational parameter $\omega = \frac{a}{b}$ we can write equality (2.8) as follows:

$$\zeta\left(s, \frac{a}{b}, \mathfrak{A}\right) = \frac{1}{k^s} \sum_{l=0}^{k-1} c_l \frac{b_l^s}{\varphi(b_l)} \sum_{\chi^{(l)} \bmod b_l} \overline{\chi^{(l)}(a_l)} L\left(s, \chi^{(l)}\right), \tag{2.9}$$

where $\frac{a+bl}{bk}=\frac{a_l}{b_l}$, $(a_l,b_l)=1$ for all $0\le l\le k-1$.

Example. Let $\mathfrak{A}=\{e^{\frac{\pi i m}{2}}:m\in\mathbb{N}_0\}$. This sequence of complex numbers is a periodic with the smallest period $k=4$ and

$$c_0=1,\quad c_1=e^{\frac{\pi i}{2}}=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i,$$

$$c_2=e^{\frac{2\pi i}{2}}=\cos\pi+i\sin\pi=-1,\quad c_3=e^{\frac{3\pi i}{2}}=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}=-i.$$

Let $\omega=\frac{2}{3}$, then

$$\frac{a_0}{b_0}=\frac{1}{6},\quad \frac{a_1}{b_1}=\frac{5}{12},\quad \frac{a_2}{b_2}=\frac{2}{3},\quad \frac{a_3}{b_3}=\frac{11}{12}.$$

From (2.8) we have

$$\zeta\left(s,\frac{2}{3},\mathfrak{A}\right)=\frac{1}{4^s}\sum_{l=0}^{3}c_l\zeta\Big(s,\frac{2}{3}\Big)$$

$$=\frac{1}{4^s}\Big(\zeta\Big(s,\frac{1}{6}\Big)+i\zeta\Big(s,\frac{5}{12}\Big)-\zeta\Big(s,\frac{2}{3}\Big)-i\zeta\Big(s,\frac{11}{12}\Big)\Big).$$

From (2.9) we obtain

$$\zeta\left(s,\frac{2}{3},\mathfrak{A}\right)=\frac{1}{4^s}\sum_{l=0}^{3}c_l\frac{b_l^s}{\varphi(b_l)}\sum_{\chi^{(l)}\bmod b_l}\overline{\chi^{(l)}(a_l)}L\left(s,\chi^{(l)}\right)$$

$$=\frac{1}{4^s}\cdot\frac{6^s}{\varphi(6)}\sum_{\chi^{(0)}\bmod 6}\overline{\chi^{(0)}(1)}L\left(s,\chi^{(0)}\right)+\frac{i}{4^s}\cdot\frac{12^s}{\varphi(12)}\sum_{\chi^{(1)}\bmod 12}\overline{\chi^{(1)}(5)}L\left(s,\chi^{(1)}\right)$$

$$-\frac{1}{4^s}\cdot\frac{3^s}{\varphi(3)}\sum_{\chi^{(2)}\bmod 3}\overline{\chi^{(2)}(2)}L\left(s,\chi^{(2)}\right)-\frac{i}{4^s}\cdot\frac{12^s}{\varphi(12)}\sum_{\chi^{(3)}\bmod 12}\overline{\chi^{(3)}(11)}L\left(s,\chi^{(3)}\right)$$

$$=\frac{3^s}{2^{s+1}}\sum_{\chi\bmod 6}\overline{\chi(1)}L\left(s,\chi\right)+\frac{3^si}{4}\sum_{\chi\bmod 12}\overline{\chi(5)}L\left(s,\chi\right)-\frac{1}{2}\cdot\frac{3^s}{4^s}\sum_{\chi\bmod 3}\overline{\chi(2)}L\left(s,\chi\right)$$

$$-\frac{3^si}{4}\sum_{\chi\bmod 12}\overline{\chi(11)}L\left(s,\chi\right).$$

2.4 Universality of Hurwitz zeta-functions and periodic Hurwitz zeta-functions

In this section we state universality theorems for Hurwitz zeta-functions. First, we present universality theorem of the classical Hurwitz zeta-functions.

Theorem 2.4.1 (Universality theorem of Hurwitz zeta-functions)**.** *Let the number ω is transcendental or rational $\neq 1, \frac{1}{2}$. Let $\mathcal{K} \subset \mathcal{D}$ be a compact subset with connected complement, and let $f(s)$ be a continuous function on $\mathcal{K}$ which is analytic in the interior of $\mathcal{K}$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0,T] : \max_{s\in\mathcal{K}} |\zeta(s+i\tau,\omega) - f(s)| < \varepsilon \right\} > 0. \tag{2.10}$$

Theorem 2.4.1 has been considered in [2] and [16] by different methods.

Next, we state the joint universality theorem for Hurwitz zeta-functions. First we define the set

$$L(\omega_1,\ldots,\omega_r) = \{\log(m+\omega_j) : m \in \mathbb{N}_0, \omega_j \in (0;1], j = 1,\ldots,r\}.$$

Theorem 2.4.2 (Joint universality theorem for Hurwitz zeta-functions [31], [41])**.** *Suppose that the set $L(\omega_1,\ldots,\omega_r)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1,\ldots,r$, let $\mathcal{K}_j \subset \mathcal{D}$ be a compact subset with connected complement, and let $f_j(s)$ be a continuous function on $\mathcal{K}_j$ which is analytic in the interior of $\mathcal{K}_j$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0,T] : \max_{1\le j\le r} \max_{s\in\mathcal{K}_j} |\zeta(s+i\tau,\omega_j) - f_j(s)| < \varepsilon \right\} > 0. \tag{2.11}$$

We recall that a generalization of the Hurwitz zeta-function is the periodic Hurwitz zeta-function $\zeta(s,\omega;\mathfrak{A})$. The universality of the function $\zeta(s,\omega;\mathfrak{A})$ with transcendental parameter ω was considered in [20] and [22]. The following statement was proved.

Theorem 2.4.3 (Javtokas and Laurinčikas [20])**.** *Let the number ω is transcendental. Let $\mathcal{K} \subset \mathcal{D}$ be a compact subset with connected complement, and let $f(s)$ be a continuous function on $\mathcal{K}$ which is analytic in the interior of $\mathcal{K}$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0,T] : \max_{s\in\mathcal{K}} |\zeta(s+i\tau,\omega;\mathfrak{A}) - f(s)| < \varepsilon \right\} > 0. \tag{2.12}$$

The joint universality theorem for periodic Hurwitz zeta-functions is proved in [30]. More about universality theorems for Hurwitz zeta-functions and periodic Hurwitz zeta-functions see in [20], [21], [31], [35], [38].

Moreover, there have been investigated *hybrid universality* (sometimes called *mixed universality*), *joint hybrid universality* (also called *joint mixed universality*) for Dirichlet L-functions and zeta-functions. More about universality of Dirichlet L-functions and other zeta-functions see in [33], [34], [37], [44], [45].

Chapter 3

Self-approximation of Hurwitz zeta-functions for transcendental parameter

3.1 Introduction

In this chapter we consider the following problem. Find all real numbers $0 < \omega \leq 1$ and d such that, for any compact subset $\mathcal{K}$ of the strip $1/2 < \sigma < 1$ and any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \max_{s\in\mathcal{K}} |\zeta(s+i\tau,\omega) - \zeta(s+id\tau,\omega)| < \varepsilon \right\} > 0. \qquad (3.1)$$

We will prove the case when ω is a transcendental number and d is a rational number. We will also show that for any transcendental number ω the inequality (3.1) is true for almost all numbers d and that for any irrational number d the inequality (3.1) is true for almost all numbers ω.

In Section 3.2 we present several facts about transcendental numbers. In Section 3.3 we investigate the set $A(d_1,\dots,d_k,\omega)$ and we prove the main results of this chapter, namely Theorem 3.4.1 and Propositions 3.4.3, 3.4.4.

3.2 Several facts about transcendental numbers

"*God made natural numbers, all else is the work of man.*" (Kronecker, cf. Weber [56].)

We recall that a *transcendental number* is a number which is not algebraic; that is, it is not the root of a non-constant polynomial equation with rational coefficients. For example, numbers e, π, e^π, $2^{\sqrt{2}}$ are transcendental. Liouville showed that number $\sum_{n=1}^{\infty} 10^{-n!}$ is transcendental, and this was one of the first numbers proven to be transcendental.

Next we present several useful facts of transcendental numbers.

- **(Hermite-Lindemann Theorem)**. For every nonzero algebraic number α, e^{α} is transcendental.
  Equivalently, if α is algebraic, $\alpha \neq 0$ and $\alpha \neq 1$, then $\log \alpha$ is transcendental.
- **(Gelfond-Schneider Theorem, 1934)**. If α and β are algebraic, $\alpha \neq 0$, $\alpha \neq 1$, β irrational, then α^{β} is transcendental.

  The next two statements are equivalent to the Gelfond-Schneider Theorem.
- If α is irrational and $\beta \neq 0$ then at least one of the numbers $\alpha, e^{\beta}, e^{\alpha\beta}$ is transcendental.
- If α is irrational and $\{\beta, \gamma\}$ is linearly independent over rationals, then at least one of the numbers $\beta, \gamma, e^{\alpha\beta}, e^{\alpha\gamma}$ is transcendental.
- **(Baker's Theorem, 1966)**. If $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers, and $\log \alpha_1, \ldots, \log \alpha_n$ is linearly independent over rationals, and $\beta_0, \ldots, \beta_n$ are algebraic and not all zero, then $\beta_0 + \sum_{j=1}^{n} \beta_j \log \alpha_j$ is transcendental.

Further results about transcendental numbers can bee found in [4], [5], [51].

3.3 The set $A(d_1, d_2, \ldots, d_k; \omega)$

In this section we define the set $A(d_1, d_2, \ldots, d_k; \omega)$ and show several properties of this set, which are interesting on their own.

Let $d_1, d_2, \ldots, d_k, \omega$ be real numbers and let ω be a real number from the interval (0,1] .
Let

$$A(d_1, d_2, \ldots, d_k; \omega) = \{d_j \log(n + \omega) : j = 1, \ldots, k; n \in \mathbb{N}_0\}$$

be a multiset. Note that in a multiset elements can appear more than once. For example, $\{1, 2\}$ and $\{1, 1, 2\}$ are different multisets, but $\{1, 2\}$ and $\{2, 1\}$ are equal multisets.

If a multiset $A(d_1, d_2, \ldots, d_k; \omega)$ is linearly independent over rational numbers, then $A(d_1, d_2, \ldots, d_k; \omega)$ is a set and the numbers $d_1, \ldots, d_k$ are linearly independent over $\mathbb{Q}$. In this thesis we work only with the set $A(d_1, d_2, \ldots, d_k; \omega)$.

Consider the case when $k = 2$. We state some examples which show possible relation between numbers d_1, d_2 and the set $A(d_1, d_2; \omega)$.
Example 1. If d_1 and d_2 are real numbers linearly dependent over $\mathbb{Q}$, then the

set $A(d_1, d_2; \omega)$ is also linearly dependent over $\mathbb{Q}$.

Example 2. Let d_1 be a rational and $d_2 = \dfrac{\log \omega}{\log(1+\omega)}$ be a real number, where ω is a transcendental number from the interval $(0;1]$. It is easy to see that numbers d_1 and d_2 are linearly independent over $\mathbb{Q}$, but the set $A(d_1, d_2; \omega)$ is linearly dependent over $\mathbb{Q}$.

Next we prove several properties of the set $A(d_1, d_2, \ldots, d_k; \omega)$ for transcendental, rational and irrational parameter ω, respectively.

Property 3.3.1. *Let $d_1 \neq 0$ be a real number and let ω be a transcendental number. Then the set $A(d_1; \omega)$ is linearly independent over $\mathbb{Q}$.*

Proof. Suppose that $d_1 \neq 0$ and there is a finite sequence of rational numbers $a_0, a_1, \ldots, a_N$ such that not all of them are equal to 0 and

$$d_1 \sum_{n=0}^{N} a_n \log(n+\omega) = 0. \tag{3.2}$$

From (3.2) we obtain

$$d_1 \log \left(\prod_{n=0}^{N} (n+\omega)^{a_n} \right) = 0$$

and

$$\prod_{n=0}^{N} (n+\omega)^{a_n} - 1 = 0. \tag{3.3}$$

Numbers $a_1, a_2, \ldots, a_N$ are rationals, then it is not difficult to see that the equality (3.3) can be written in the form $P(\omega) = 0$, where $P(\omega)$ is a polynomial. But ω is a transcendental number, and we obtain contradiction. This gives that the set $A(d_1; \omega)$ is linearly independent over $\mathbb{Q}$. $\square$

Property 3.3.2. *Let $d_1, d_2, \ldots, d_k$ be real numbers and ω, $0 < \omega \leq 1$ be a rational number, then the set $A(d_1, d_2, \ldots, d_k; \omega)$ is linearly dependent over $\mathbb{Q}$.*

Proof. It is enough to consider the case when numbers $d_1, d_2, \ldots, d_k$ are linearly independent over $\mathbb{Q}$. Let $\omega = \frac{a}{b}$, where $(a,b) = 1$. The set $A(d_1, d_2, \ldots, d_k; \omega)$ consists of elements

$$d_1 \log \frac{a}{b}; \quad d_1 \log \left(1 + \frac{a}{b}\right); d_1 \log \left(2 + \frac{a}{b}\right); \ldots\ldots; d_1 \log \left(r + \frac{a}{b}\right)$$

$$d_2 \log \frac{a}{b}; d_2 \log \left(1 + \frac{a}{b}\right); d_2 \log \left(2 + \frac{a}{b}\right); \ldots\ldots; d_2 \log \left(r + \frac{a}{b}\right)$$

. . . .

$$d_k \log \frac{a}{b}; d_k \log \left(1 + \frac{a}{b}\right); d_1 \log \left(2 + \frac{a}{b}\right); \ldots\ldots; d_k \log \left(r + \frac{a}{b}\right),$$

where $r \to \infty$.

Suppose that there exists a finite sequence of rational numbers a_{in}, $i = 1, \ldots, k$, $n = 0, \ldots, N$ such that not all of these numbers are equal to 0 and equality

$$d_1 \sum_{n=0}^{N} a_{1n} \log \left(n + \frac{a}{b}\right) + \cdots + d_k \sum_{n=0}^{N} a_{kn} \log \left(n + \frac{a}{b}\right) = 0 \tag{3.4}$$

is valid.

Consider the set of numbers

$$d_1 a_{10} \log \frac{a}{b}; d_1 a_{11} \log \left(1 + \frac{a}{b}\right); d_1 a_{12} \log \left(2 + \frac{a}{b}\right); \ldots\ldots; d_1 a_{1t} \log \left(t + \frac{a}{b}\right)$$

$$d_2 a_{20} \log \frac{a}{b}; d_2 a_{21} \log \left(1 + \frac{a}{b}\right); d_2\ a_{22} \log \left(2 + \frac{a}{b}\right); \ldots\ldots; d_2 a_{2t} \log \left(t + \frac{a}{b}\right),$$

where $t > 2$.
Take $a_{10} = a_{20} = a_{1t} = a_{2t} = 1$, $a_{11} = a_{21} = a_{12} = a_{22} = -1$, and let other $a_{ij} = 0$, and let (3.4) is valid.

Next we prove that there is the integer number $t = t(a, b)$, $t > 2$ such that the equality (3.4) is valid. Numbers d_1, d_2 are linearly independent over $\mathbb{Q}$. Thus we have

$$\log \left(\frac{a}{b}\right) \left(\frac{b}{a+b}\right) \left(\frac{b}{a+2b}\right) \left(\frac{a+tb}{b}\right) = 0. \tag{3.5}$$

From the last equation we obtain

$$a(bt + a) = (b + a)(b + a)$$

and

$$t = \frac{3a + 2b}{a}.$$

This implies Property 3.3.2. $\square$

Also for the proof of Property 3.3.2 we can use identity

$$\left(t + \frac{a}{b}\right)\left((t+1)(b+1) + a + \frac{a}{b}\right) = \left(t + 1 + \frac{a}{b}\right)\left(t(b+1) + a + \frac{a}{b}\right)$$

(see [10]). Alternatively, one can apply results given by Pólya and Szegö (see [46], Chapter 8, Problems 95 and 109).

Property 3.3.3. *Let $d_1, d_2, \dots, d_k$ be real numbers and ω, $0 < \omega \le 1$ be an irrational number of the form $\omega = \sqrt{r} - q$, where $r, q \in \mathbb{N}$ and $r - q^2 = 1$. Then the set $A(d_1, d_2, \dots, d_k; \omega)$ is linearly dependent over $\mathbb{Q}$.*

Proof. Consider the set

$$A(d_1, d_2, \dots, d_k; \omega) = \{d_j \log(n + \omega) : j = 1, \dots, k; n \in \mathbb{N}_0\}.$$

It is easy to see that this set contains elements $d_j \log(\sqrt{r} - q)$ and $d_j \log(\sqrt{r} + q)$, where $j = 1, 2, \dots, k$. This implies Property 3.3.3. □

We recall that ω is an algebraic integer of degree 2, if $\omega^2 = -a\omega - b$, where $a, b \in \mathbb{Z}$. The following property generalizes Property 3.3.3.

Property 3.3.4. *Let $d_1, d_2, \dots, d_k$ be real numbers and ω be an algebraic integer of degree 2. Then the set $A(d_1, d_2, \dots, d_k; \omega)$ is linearly dependent over $\mathbb{Q}$.*

Proof. See Theorem 1 in [10]. □

3.4 Main Theorem 3.4.1 and useful propositions

In this chapter we will prove the following theorem, which can be called the *self-approximation theorem of Hurwitz zeta-function with transcendental parameter.*

Theorem 3.4.1. *Let $l \le m$ be positive integers and let ω be a transcendental number from the interval (0,1]. Let $d_1, \dots, d_l \in \mathbb{R}$ be such that the set*

$$A(d_1, d_2, \dots, d_l; \omega) = \{d_j \log(n + \omega) : j = 1, \dots, l; n \in \mathbb{N}_0\}$$

is linearly independent over $\mathbb{Q}$. For $m > l$, let $d_{l+1}, \dots, d_m \in \mathbb{R}$ be such that each d_k, $k = l + 1, \dots, m$ is a linear combination of $d_1, \dots, d_l$ over $\mathbb{Q}$.

Then, for any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} meas \Big\{\tau \in [0, T] : \max_{1\le j,k\le m} \max_{s\in\mathcal{K}} |\zeta(s + id_j\tau, \omega) - \zeta(s + id_k\tau, \omega)| < \varepsilon\Big\} > 0. \tag{3.6}$$

In the inequality (3.6), for almost all ε, 'lim inf' can be replaced by 'lim' similarly as in Theorem 2 of [12].

Next we recall the definition of a countable set.

Definition 3.4.2. *A set S is countable if there exists an injective function*

$$f : S \to \mathbb{N}.$$

For example, the set of all integer numbers is countable, but the set of all transcendental numbers is uncountable. More about countable sets can be found in [50] (see Chapter 2).

The following propositions show that for any positive integer l 'most' collections of real numbers $d_1, d_2, \dots, d_l, \omega$, where $0 < \omega \leq 1$, are such that the set $A(d_1, d_2, \dots, d_l; \omega)$ is linearly independent over $\mathbb{Q}$.

Proposition 3.4.3. *Let ω be a transcendental number and $l \geq 2$. If $A(d_1, d_2, \dots, d_{l-1}; \omega)$ is linearly independent over $\mathbb{Q}$, then the set*

$$E = \{d_l \in \mathbb{R} : A(d_1, d_2, \dots, d_l; \omega) \text{ is linearly dependent over } \mathbb{Q}\}$$

is countable.

Proposition 3.4.4. *Let $d_1, d_2, \dots, d_l$ be real numbers linearly independent over $\mathbb{Q}$. Then the set*

$$H = \{\omega \in (0, 1] : A(d_1, d_2, \dots, d_l; \omega) \text{ is linearly dependent over } \mathbb{Q}\}$$

is countable.

In Section 3.6 we will prove Theorem 3.4.1. Section 3.7 is devoted to proofs of Propositions 3.4.3 and 3.4.4.

It should be mentioned that it is difficult to construct an example, where $d_1, d_2, \dots, d_l$ are linearly independent over $\mathbb{Q}$ and $A(d_1, d_2, \dots, d_l; \omega)$ is also are linearly independent over $\mathbb{Q}$.

3.5 Auxiliary lemmas

We start from the lemmas which will be useful in the proof of the main theorem. We recall some definitions.

Let U be an open bounded rectangle with vertices on the lines $\sigma = \sigma_1$ and $\sigma = \sigma_2$, where $1/2 < \sigma_1 < \sigma_2 < 1$.

Lemma 3.5.1. *Let $\mathcal{K}$ be a compact subset of the rectangle U and let*

$$d = \min_{z \in \partial U} \min_{s \in \mathcal{K}} |s - z|.$$

If $f(s)$ is analytic on U and

$$\int_U |f(s)|^2 \, d\sigma dt \leq \varepsilon,$$

then

$$\max_{s \in \mathcal{K}} |f(s)| \leq \frac{\sqrt{\varepsilon/\pi}}{d}.$$

Proof. The lemma above can be found in [16] (see Lemma 2.5). $\square$

Definition 3.5.2. *Let $\overline{x} \in \mathbb{R}^N$ and $\gamma \subset \mathbb{R}^N$. The notation $\overline{x} \in \gamma \mod 1$ means that there exists an integer vector $\overline{y}$ in $\mathbb{R}^N$ such that $\overline{x} - \overline{y} \in \gamma$.*

Next we recall the notation of the Jordan volume of the region $\gamma \subset \mathbb{R}^N$. Consider the sets of parallelepipeds γ_1 and γ_2, with sides parallel to the axes and of volume V_1 and V_2 with $\gamma_1 \subset \gamma \subset \gamma_2$. If there are γ_1 and γ_2 such that $\limsup_{\gamma_1} V_1$ coincides with $\liminf_{\gamma_2} V_2$, then γ has *Jordan volume*

$$V = \limsup_{\gamma_1} V_1 = \limsup_{\gamma_2} V_2.$$

If the Jordan volume exists, it is also defined in the sense of Lebesgue and equal to it.

Next we state the generalized Kronecker's theorem.

Lemma 3.5.3. *Let $a_1, \ldots, a_N$ be real numbers linearly independent over the rational numbers. Let γ be a region of the N-dimensional unit cube with volume V (in the Jordan sense). Let $I_\gamma(T)$ be the sum of the intervals between $t = 0$ and $t = T$ for which the point $(a_1 t, \ldots, a_N t)$ is $\mod 1$ inside γ. Then*

$$\lim_{T \to \infty} \frac{I_\gamma(T)}{T} = V.$$

Proof. This is Theorem 1 in Apendix, Section 8, of [23]. $\square$

For a curve $\gamma(t)$ in $\mathbb{R}^N$ we introduce the notation

$$\{\gamma(t)\} = (\gamma_1(t) - [\gamma_1(t)], \ldots, \gamma_N(t) - [\gamma_N(t)]),$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

Definition 3.5.4. *Let $\gamma(t)$ be a continuous function with domain of definition $(0, \infty]$ and range $\mathbb{R}^N$. We say that the curve $\gamma(t)$ is uniformly distributed $\mod 1$ in $\mathbb{R}^N$ if the following relation holds for every parallelepiped*

$$\Pi = [\alpha_1, \beta_1] \times \cdots \times [\alpha_N, \beta_N], \;\; 0 \leq \alpha_j < \beta_j \leq 1, \;\; \textit{for } j = 1, \ldots, N:$$

$$\lim_{T\to\infty}\frac{1}{T}meas\Big\{t : t\in[0,T], \gamma(t)\in\Pi \mod 1\Big\} = \prod_{j=1}^{N}\Big(\beta_j-\alpha_j\Big).$$

Lemma 3.5.5. *Suppose that the curve $\gamma(t)$ is uniformly distributed $\mod 1$ in $\mathbb{R}^N$. Let D be a closed and Jordan measurable subregion of the unit cube in $\mathbb{R}^N$ and let Ω be a family of complex-valued continuous functions defined on D. If Ω is uniformly bounded and equicontinuous, then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f\left(\{\gamma(t)\}\right)1_D(t)dt = \int_D f(x_1,\ldots,x_N)dx_1\ldots dx_N$$

uniformly with respect to $f\in\Omega$, where $1_D(t)$ is equal to 1 if $\gamma(t)\in D \mod 1$, and 0 otherwise.

Proof. This lemma is Theorem 3 in Appendix, Section 8, of [23]. □

We recall that the Lerch zeta-function $L(\lambda,\omega,s)$, for $\sigma>1$, is defined by

$$L(\lambda,\omega,s)=\sum_{m=0}^{\infty}\frac{e^{2\pi i\lambda m}}{(m+\omega)^s}.$$

Here $\omega,\lambda\in\mathbb{R}$, $0<\omega\le 1$, are fixed parameters. For $\lambda\in\mathbb{Z}$ the Lerch zeta-function $L(\lambda,\omega,s)$ reduces to the Hurwitz zeta-function $\zeta(s,\omega)$.

Now we state a mean square value theorem of Lerch zeta-function, see [14].

Lemma 3.5.6. *For $0<\lambda,\alpha\le 1$ we have, as T turns to infinity,*

$$\int_1^T |L(\lambda,\alpha,1/2+it)|^2dt = T\log\frac{T}{2\pi}+T(c(\alpha)+c(\lambda)-1)+O(T^{\frac{1}{2}}\log T)$$

and for $\frac{1}{2}<\sigma<1$,

$$\int_1^T |L(\lambda,\alpha,\sigma+it)|^2dt = \zeta(2\sigma,\alpha)T+\frac{(2\pi)^{2\sigma-1}}{2-2\sigma}\zeta(2-2\sigma,\lambda)T^{2-2\sigma}$$

$$+O(T^{1-\sigma}\log T+T^{\frac{\sigma}{2}}).$$

For $\sigma=1/2$, Rane [48] proved a mean square formula for Hurwitz zeta-functions with the same error term $O(T^{1-\sigma}\log T+T^{\frac{\sigma}{2}})$.

We recall that $f(s)$ is a function of finite order if

$$f(s)\ll_\sigma |t|^{A(\sigma)},\quad \text{for } \sigma>1, |t|\to\infty.$$

Further we state Carlson theorem (see [8] and [53]).

Lemma 3.5.7. *If $f(s) = \sum_{n=0}^{\infty} \frac{a_n}{n^s}$ is regular and of finite order for $\sigma \geq \alpha$, and*

$$\frac{1}{2T} \int_{-T}^{T} |f(\alpha + it)|^2 dt$$

is bounded as $T \to \infty$, then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\alpha + it)|^2 dt = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}}$$

for $\sigma > \alpha$, and uniformly in any strip $\alpha < \sigma_1 \leq \sigma \leq \sigma_2$.

3.6 Proof of Theorem 3.4.1

We follow the proof of Theorem 1 in [12]. As it was already mentioned, the proof of Theorem 1 in [12] contains a gap; however, here we avoid this gap because we work directly with $\zeta(s, \omega)$ instead of $\log \zeta(s, \omega)$. Let us start with a truncated Hurwitz zeta-function

$$\zeta_v(s, \omega) = \sum_{q \leq v} \frac{1}{(q + \omega)^s}.$$

By conditions of the theorem, there are integers $a \neq 0$ and $a_{k,1}, a_{k,2}, \ldots, a_{k,l}$ such that

$$d_k = \frac{1}{a}(a_{k,1}d_1 + a_{k,2}d_2 + \cdots + a_{k,l}d_l) \quad \text{for} \quad l < k \leq m. \tag{3.7}$$

Let

$$A = \max_{l<k\leq m} \{|a_{k,1}| + |a_{k,2}| + \cdots + |a_{k,l}|\}.$$

If

$$\left\| \tau \frac{d_n \log(q + \omega)}{2\pi a} \right\| < \delta \quad \text{for} \quad q \leq v \text{ and } 1 \leq n \leq l \tag{3.8}$$

then, by the relation (3.7),

$$\left\| \tau \frac{d_k \log(q + \omega)}{2\pi} \right\| < A\delta \quad \text{for} \quad q \leq v \text{ and } l < k \leq m.$$

By this and by the continuity in s of the function $\zeta_v(s, \omega)$, we find that for any $\varepsilon > 0$ there is $\delta > 0$ such that for τ satisfying (3.8)

$$\max_{1\leq k,n\leq m} \max_{s\in\mathcal{K}} |\zeta_v(s + id_k\tau, \omega) - \zeta_v(s + id_n\tau, \omega)| < \varepsilon. \tag{3.9}$$

For positive numbers δ, v, and T we define the set $S_T = S_T(\delta, v)$ by

$$S_T = \left\{ \tau : \tau \in [0,T], \left\| \tau \frac{d_n \log(q+\omega)}{2\pi a} \right\| < \delta, \ q \le v, \ 1 \le n \le l \right\}. \tag{3.10}$$

Let U be an open bounded rectangle with vertices on the lines $\sigma = \sigma_1$ and $\sigma = \sigma_2$, where $1/2 < \sigma_1 < \sigma_2 < 1$, such that the set $\mathcal{K}$ is in U. Let $p > v$ be a positive integers. We have

$$\begin{aligned} &\frac{1}{T} \int\limits_{S_T} \int\limits_{U} \sum_{k=1}^{m} |\zeta_p(s + id_k\tau, \omega) - \zeta_v(s + id_k\tau, \omega)|^2 \, d\sigma dt d\tau \\ &= \sum_{k=1}^{m} \int\limits_{U} \frac{1}{T} \int\limits_{S_T} |\zeta_p(s + id_k\tau, \omega) - \zeta_v(s + id_k\tau, \omega)|^2 \, d\tau d\sigma dt. \end{aligned}$$

To evaluate the inner integrals of the right-hand side of the last equality we will apply Lemma 3.5.5. By generalized Kronecker's theorem 3.5.3 and by linear independence of $A(d_1, d_2, \ldots, d_l; \omega)$ the curve

$$\omega(\tau) = \left(\tau \frac{d_k \log(q+\omega)}{2\pi a} \right)_{0 \le q \le p}^{1 \le k \le l}$$

is uniformly distributed $\mod 1$ in $\mathbb{R}^{l(p+1)}$. Let R' be a subregion of the $l(p+1)$-dimensional unit cube defined by inequalities

$$\|y_{k,q}\| \le \delta \quad \text{for} \quad 1 \le k \le l \text{ and } 0 \le q \le v$$

and

$$\left| y_{k,q} - \frac{1}{2} \right| \le \frac{1}{2} \quad \text{for} \quad 1 \le k \le l \text{ and } v+1 \le q \le p.$$

Let R be a subregion of the $l(v+1)$-dimensional unit cube defined by inequalities

$$\|y_{k,q}\| \le \delta \quad \text{for} \quad 1 \le k \le l \text{ and } 0 \le q \le v$$

Clearly $\operatorname{meas} R' = \operatorname{meas} R = (2\delta)^{l(v+1)}$. Let

$$\zeta_{p,v}(s + id_k\tau, \omega) = \zeta_p(s + id_k\tau, \alpha) - \zeta_v(s + id_k\tau, \omega). \tag{3.11}$$

Then in view of the linear dependence (3.7) we get

$$
\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_{S_T}\sum_{k=1}^{m}|\zeta_{p,v}(s+id_k\tau,\omega)|^2\,d\tau\\
&=\lim_{T\to\infty}\frac{1}{T}\int_{S_T}\left(\sum_{k=1}^{l}|\zeta_{p,v}(s+id_k\tau,\omega)|^2\right.\\
&\quad\left.+\sum_{k=l+1}^{m}\left|\zeta_{p,v}\left(s+\frac{i}{a}(a_{k,1}d_1+a_{k,2}d_2+\cdots+a_{k,l}d_l)\tau,\omega\right)\right|^2\right)d\tau.
\end{aligned}
$$

By Lemma 3.5.5 and equality (3.11) we obtain that the last limit is equal to

$$
\begin{aligned}
&\int_{R'}\left(\sum_{k=1}^{l}\left|\sum_{v<q\le p}\frac{e^{-2\pi i a y_{k,q}}}{(q+\omega)^s}\right|^2\right.\\
&\quad\left.+\sum_{k=l+1}^{m}\left|\sum_{v<q\le p}\frac{e^{-2\pi i(a_{k,1}y_{1,q}+a_{k,2}y_{2,q}+\cdots+a_{k,l}y_{l,q})}}{(q+\omega)^s}\right|^2\right)dy_{1,1}\dots dy_{l,p}
\end{aligned}
$$

$$
\begin{aligned}
&=\operatorname{meas}R\int_0^1\dots\int_0^1\left(\sum_{k=1}^{l}\left|\sum_{v<q\le p}\frac{e^{-2\pi i y_{k,q}}}{(q+\omega)^s}\right|^2\right.\\
&\quad\left.+\sum_{k=l+1}^{m}\left|\sum_{v<q\le p}\frac{e^{-2\pi i(a_{k,1}y_{1,q}+a_{k,2}y_{2,q}+\cdots+a_{k,l}y_{l,q})}}{(q+\omega)^s}\right|^2\right)dy_{1,v+1}\dots dy_{l,p}\\
&=m\operatorname{meas}R\sum_{v<q\le p}\frac{1}{(q+\omega)^{2\sigma}}\ll\operatorname{meas}R\sum_{q>v}\frac{1}{(q+\omega)^{2\sigma}}.
\end{aligned}
$$

Remark 3.6.1. We use notations of a big O and $\ll$ interchangeably to describe the limiting behavior of a function when its variable tends towards infinity. We write

$$f(x)=O(g(x)) \text{ or } f(x)\ll g(x)$$

if and only if there a positive number c and real number x_0 such that

$$|f(x)|\le c|g(x)|,\ \text{for all } x>x_0.$$

Consequently,

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_{S_T}\int\limits_{U}\sum_{k=1}^{m}|\zeta_p(s+id_k\tau,\omega)-\zeta_v(s+id_k\tau,\omega)|^2\,d\sigma dt d\tau \tag{3.12}$$
$$\ll \operatorname{meas} R\sum_{q>v}\frac{1}{(q+\omega)^{2\sigma_1}}.$$

Again, by Lemma 3.5.3,

$$\lim_{T\to\infty}\frac{1}{T}\operatorname{meas} S_T=\operatorname{meas} R. \tag{3.13}$$

By (3.12) and (3.13), for large v, as $T\to\infty$, we have

$$\operatorname{meas}\left\{\tau:\tau\in S_T,\int_U\sum_{k=1}^{m}|\zeta_{p,v}(s+id_k\tau,\omega)|^2\,d\sigma dt<\sqrt{\sum_{q>v}\frac{1}{(q+\omega)^{2\sigma_1}}}\right\}$$
$$>\frac{1}{2}T\operatorname{meas} R.$$

Then Lemma 3.5.1 gives

$$\operatorname{meas}\left\{\tau:\tau\in S_T,\max_{s\in\mathcal{K}}\sum_{k=1}^{m}|\zeta_{p,v}(s+id_k\tau,\omega)|\le\frac{m}{d\sqrt{\pi}}\left(\sum_{q>v}\frac{1}{(q+\omega)^{2\sigma_1}}\right)^{\frac{1}{4}}\right\}$$
$$>\frac{1}{2}T\operatorname{meas} R,$$

where $d=\min_{z\in\partial U}\min_{s\in\mathcal{K}}|s-z|$. Therefore, we obtain that for any $\varepsilon>0$ there is $v=v(\varepsilon)$ such that for any $p>v$

$$\operatorname{meas}\left\{\tau:\tau\in S_T,\max_{s\in\mathcal{K}}\sum_{k=1}^{m}|\zeta_p(s+id_k\tau,\omega)-\zeta_v(s+id_k\tau,\omega)|<\varepsilon\right\} \tag{3.14}$$
$$>\frac{1}{2}T\operatorname{meas} R.$$

Now we will prove that for any $\delta>0$ there is $p=p(\delta)$ such that

$$\operatorname{meas}\left\{\tau:\max_{s\in\mathcal{K}}\sum_{k=1}^{m}|\zeta(s+id_k\tau,\omega)-\zeta_p(s+id_k\tau,\omega)|<\delta\right\} \tag{3.15}$$
$$>(1-\delta)T.$$

The last formula together with (3.9), (3.10) and (3.14) yields Theorem 3.4.1. We return to the proof of (3.15). By Lemma 3.5.6 and by Lemma 3.5.7 we obtain

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\zeta(s+ix\tau,\omega) - \zeta_p(s+ix\tau,\omega)|^2 \, d\tau = \sum_{q>p} \frac{1}{(q+\omega)^{2\sigma}},$$

where x is fixed. Thus (3.15) follows in view of

$$\int_0^T \int_U \sum_{k=1}^m |\zeta(s+id_k\tau,\omega) - \zeta_p(s+ix\tau,\omega)|^2 \, d\sigma dt d\tau \ll T \sum_{q>p} \frac{1}{(q+\omega)^{2\sigma_1}}.$$

Theorem 3.4.1 is proved.

3.7 Proofs of Propositions 3.4.3 and 3.4.4

Proof of Proposition 3.4.3. Let Ψ be a set of all rational numbers sequences, where each sequence has only finitely many nonzero elements. Then Ψ is a countable set. By $\mathbf{0}$ we denote the sequence all elements of which are zeros. Let $d_1 = 1$. Recall that the set $A(1;\omega)$ is linearly independent. Then in view of the linear independence of $A(d_1, d_2, \dots, d_{l-1}; \omega)$ we obtain that the set

$$E = \Bigg\{ -\frac{d_1 \sum_{n=0}^{\infty} a_{1n} \log(n+\omega) + \dots + d_{l-1} \sum_{n=0}^{\infty} a_{l-1n} \log(n+\omega)}{\sum_{n=0}^{\infty} a_{ln} \log(n+\omega)} : \\ (a_{10}, a_{11}, \dots, a_{(l-1)0}, a_{(l-1)1}, \dots, a_{l0}, a_{l1}, \dots) \in \Psi \setminus \mathbf{0}, \\ (a_{l0}, a_{l1}, \dots) \neq \mathbf{0} \Bigg\}$$

is a countable. This proves the proposition. □

Proof of Proposition 3.4.4. We use the same notations as in the proof of Proposition 3.4.3. Similarly as before we put

$$H = \Bigg\{ \omega \in I : d_1 \sum_{n=0}^{\infty} a_{1n} \log(n+\omega) + \dots + d_l \sum_{n=0}^{\infty} a_{ln} \log(n+\omega) = 0, \\ (a_{10}, a_{11}, \dots, a_{20}, a_{21}, \dots, \dots, a_{l0}, a_{l1}, \dots) \in \Psi \setminus \mathbf{0} \Bigg\}.$$

Recall that Ψ is a countable set. If, for a fixed

$$(a_{10}, a_{11}, \dots, a_{20}, a_{21}, \dots, \dots, a_{l0}, a_{l1}, \dots) \in \Psi \setminus \mathbf{0},$$

the function

$$f(\omega) = d_1 \sum_{n=0}^{\infty} a_{1n} \log(n+\omega) + \cdots + d_l \sum_{n=0}^{\infty} a_{ln} \log(n+\omega)$$

has only finite number of zeros in $(0,1]$, then the set H is countable. Thus to prove the proposition it remains to show that $f(\omega)$ has finitely many zeros in the interval $(0,1]$. In view of the condition that $d_1, d_2, \ldots, d_l$ are linearly independent and by the definition of Ψ, we have that there is a finite collection of real numbers $b_0, b_1, \ldots, b_m$ such that $b_m \neq 0$ and

$$f(\omega) = b_0 \log(\omega) + b_1 \log(1+\omega) + \cdots + b_m \log(m+\omega).$$

Let b_n, $n \leq m$ be the first coefficient not equal to zero. Then we see that $f(\omega)$ is unbounded in $(-n, 1/2)$ and is bounded in $(1/2, 1]$. Thus $f(\omega)$ is not a constant in $(-n, 1]$. Moreover, there is a small positive number ω_0 such that $f(\omega) \neq 0$ if $\omega \in (-n, -n+\omega_0)$. We consider $f(\omega)$ as an analytic function in the half-plane $\Re\omega > -n$ of the complex plane. A set of zeros of a non-constant analytic function is discrete. Thus there are finitely many zeros in the disc $|1-\omega| \leq 1+n-\omega_0$. We obtained that the function $f(\omega)$ has finitely many zeros in $(0,1]$. This proves the proposition. $\square$

Chapter 4

Self-approximation of Hurwitz zeta-functions for rational parameter

4.1 Main Theorem 4.1.1

In this chapter, we show the self-approximation property for Hurwitz zeta-functions with rational parameters. Namely, we prove that $\zeta(s+i\alpha\tau, \frac{a}{b})$ approximates uniformly $\zeta(s+i\beta\tau, \frac{a}{b})$ for infinitely many real τ, where α, β are arbitrary real numbers linearly independent over $\mathbb{Q}$, and s is in a compact set lying in the open right half of the critical strip.

We recall that for rational $\omega = \frac{a}{b}$ satisfying $0 < a < b$ and $gcd(a,b) = 1$ the Hurwitz zeta function might be expressed as a linear combination of Dirichlet L-functions:

$$\zeta\left(s+i\tau, \frac{a}{b}\right) = \frac{b^{s+i\tau}}{\varphi(b)} \sum_{\chi \bmod b} \overline{\chi(a)} L(s+i\tau, \chi). \tag{4.1}$$

More precisely, we use (4.1) to prove the following theorem, which can be called the *self-approximation theorem of Hurwitz zeta-function with rational parameter.*

Theorem 4.1.1. *Let $\omega = \frac{a}{b}$ be a rational number satisfying $0 < a < b$ and $gcd(a,b) = 1$. Moreover, suppose that α, β are real numbers linearly independent over $\mathbb{Q}$, and $\mathcal{K}$ is any compact subset of the strip $1/2 < \sigma < 1$.*

Then, for any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} meas\left\{\tau \in [0,T] : \max_{s\in\mathcal{K}} \left|\zeta\left(s+i\alpha\tau, \frac{a}{b}\right) - \zeta\left(s+i\beta\tau, \frac{a}{b}\right)\right| < \varepsilon\right\} > 0.$$

4.2 Preliminaries

In order to prove our main theorem we need some results about linear independence of the set

$$\left\{\alpha\frac{\log p}{2\pi}\right\}_{p\in\mathbb{P}} \cup \left\{\beta\frac{\log p}{2\pi}\right\}_{p\in\mathbb{P}},$$

where α, β are real numbers linearly independent over $\mathbb{Q}$, and $\mathbb{P}$ denotes the set of all rational primes.

Lemma 4.2.1. *For arbitrary real numbers α, β linearly independent over $\mathbb{Q}$, there exists a finite set of primes $A = A(\alpha, \beta)$ containing at most two elements such that the following set*

$$\{\alpha \log p\}_{p\in\mathbb{P}\setminus A} \cup \{\beta \log p\}_{p\in\mathbb{P}} \tag{4.2}$$

is linearly independent over $\mathbb{Q}$.

Proof. This is Lemma 2.4 in [43]. □

Lemma 4.2.2. *Suppose that $(a_n)_{n=1}^{\infty}$ are real numbers linearly independent over $\mathbb{Q}$. Moreover, assume that $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$ and so $\theta_1, \dots, \theta_m \in \mathbb{R}$. Then there exist finite sets $J \subset \{1, 2, \dots, m\}$ and $A = A(\alpha_1, \dots, \alpha_m) \subset \mathbb{Z}_+$ such that the set*

$$\{a_i\}_{i\in A\cup M} \cup \{\alpha_i\}_{i\in J}$$

is linearly independent over $\mathbb{Q}$ for every finite set of non-negative integers M with $M \cap A = \emptyset$.
Moreover, there exist real numbers θ_i^*, $i \in A$ and a positive integer N such that

$$\max_{i\notin J} \|N\tau\alpha_i - \theta_i\| < \varepsilon,$$

whenever the following inequalities hold

$$\max_{i\in J}\left\|\tau\alpha_i - \frac{\theta_i}{N}\right\| < \frac{\varepsilon}{N},$$

$$\max_{i\in A}\left\|\tau a_i - \frac{\theta_i^*}{N}\right\| < \frac{\varepsilon}{N}.$$

Proof. This is Corollary 2.7 in [43]. □

The linear independence over $\mathbb{Q}$ allows us to apply the following classical Kronecker theorem.

Lemma 4.2.3. *(Kronecker's theorem). For $x \in \mathbb{R}$, let $\|x\|$ denote the distance from x to the nearest integer. Then for arbitrary real numbers $\alpha_1, \ldots, \alpha_n$ linearly independent over $\mathbb{Q}$, any real numbers $\theta_1, \ldots, \theta_n$ and any numbers $\varepsilon_1, \ldots, \varepsilon_n$, the set of τ such that*

$$\|\tau\alpha_i - \theta_i\| < \varepsilon_i, \qquad \textit{for all } 1 \le i \le n \tag{4.3}$$

has a positive density, which is equal to $2^n \prod_{1 \le i \le n} \varepsilon_i$.

For the sake of simplicity, let $\Omega = \prod_p \mathbb{R}$ denote the set of all sequences of real numbers $\Theta = (\theta_p)_p$ indexed by prime numbers. Moreover, for any finite set $M \subset \mathbb{P}$ and any Dirichlet character χ, we put

$$L_M(s, \chi, \Theta) = \prod_{p \in M} \left(1 - \frac{\chi(p)e(-\theta_p)}{p^s}\right)^{-1}$$

and

$$L(s, \chi)_{|M} = L(s, \chi) \prod_{p \in M} \left(1 - \frac{\chi(p)}{p^s}\right),$$

where $\sigma > \frac{1}{2}$.

Now, let us recall the property for Dirichlet L-functions associated to pairwise non-equivalent Dirichlet characters, which plays a crucial role in the proof of our main result. Following [26], we call an open and bounded subset G of $\mathbb{C}$ *admissible*, when for every $\varepsilon > 0$ the set

$$G_\varepsilon = \left\{ s \in \mathbb{C} : |s - w| < \varepsilon \text{ for certain } w \in G \right\}$$

has connected complement.

Lemma 4.2.4. *Let $\chi_1, \ldots, \chi_n$ be pairwise non-equivalent Dirichlet characters and admissible domain G be an admissible domain such that*

$$\overline{G} \subset \mathcal{D}, \mathcal{D} := \{s \in \mathbb{C} : 1/2 < \sigma < 1\}.$$

Moreover, assume that $f_1, \ldots, f_n$ are analytic and non-vanishing functions on the closure $\overline{G}$. Then, for every finite set M of primes, there exists a sequence of finite sets $M_1 \subset M_2 \subset \ldots \subset \mathbb{P}$ such that

$$\bigcup_{k=1}^{\infty} M_k = \{p : p \notin M\}$$

and for certain $\Theta_k \in \Omega$ *as* $k \to \infty$

$$L_{M_k}(s, \chi_j, \Theta_k) \to f_j(s) \quad \textit{uniformly for } s \in \overline{G}, j = 1, \dots, n.$$

Proof. See Remark 2.1 and the acceptability property in [44]. $\square$

4.3 An important Lemma 4.3.1 and auxiliary lemmas

In this section we present the following main lemma, which will be useful in the proof of Theorem 4.1.1.

Lemma 4.3.1. *Let* $\chi_1, \dots, \chi_n$ *be pairwise non-equivalent Dirichlet characters,* G *be any admissible set such that* $\overline{G} \subset \mathcal{D}$ *and functions* f_j, g_j $(j = 1, \dots, n)$ *are analytic and non-vanishing on* $\overline{G}$*. Moreover, suppose that* B *is a finite set of primes,* α*,* β *are real numbers linearly independent over* $\mathbb{Q}$*, and the set* A *has the same meaning as in Lemma 4.2.1.*

Then, for every $\varepsilon > 0$ *and an arbitrary set* $G_0 \subset \overline{G_0} \subset G$*, there exist finite sets*

$$A_1 \subset \mathbb{P} \setminus (A \cup B), A_2 \subset \mathbb{P} \setminus B$$

and real numbers

$$\theta_p^{(1)}, p \in A_1, \theta_p^{(2)}, p \in A_2$$

such that the set of real numbers τ *satisfying the following inequalities*

$$\max_{1 \le j \le n} \max_{s \in \overline{G_0}} \Bigg| L(s + i\alpha\tau, \chi_j)_{|(A \cup A_1 \cup B)} - f_j(s) \prod_{p \in A \cup B} \left(1 - \frac{\chi_j(p)}{p^s}\right) \prod_{p \in A_1} \left(1 - \frac{\chi_j(p) e(-\theta_p^{(1)})}{p^s}\right) \Bigg| < \varepsilon,$$

$$\max_{1 \le j \le n} \max_{s \in \overline{G_0}} \Bigg| L(s + i\beta\tau, \chi_j)_{|(A_2 \cup B)} - g_j(s) \prod_{p \in B} \left(1 - \frac{\chi_j(p)}{p^s}\right) \prod_{p \in A_2} \left(1 - \frac{\chi_j(p) e(-\theta_p^{(2)})}{p^s}\right) \Bigg| < \varepsilon,$$

$$\max_{\gamma \in \{\alpha, \beta\}} \max_{p \in B} \left\| \gamma\tau \frac{\log p}{2\pi} \right\| < \varepsilon,$$

$$\max_{p \in A} \left\| \alpha\tau \frac{\log p}{2\pi} \right\| < \varepsilon,$$

$$\max_{p\in A_1}\left\|\alpha\tau\frac{\log p}{2\pi}-\theta_p^{(1)}\right\|<\varepsilon,$$

$$\max_{p\in A_2}\left\|\beta\tau\frac{\log p}{2\pi}-\theta_p^{(2)}\right\|<\varepsilon$$

has a positive lower density.

Next we present several lemmas, which helps to prove main Lemma 4.3.1.

Definition 4.3.2. *If M is a finite set of primes, $s\in\mathbb{C}$, and $\overline{\theta}\in\Omega$, then we set*

$$\zeta_M(s,\overline{\theta})=\prod_{p\in M}(1-e^{-2\pi i\theta_p}/p^s)^{-1}.$$

Lemma 4.3.3. *Let $0<r<\frac{1}{4}$. Suppose that $g(s)$ is analytic for $|s|<r$ and continuous for $s\le r$. Then for any $\varepsilon>0$ and any $y>0$ there exists a finite set M such that*

$$\{p:p\le y\}\subset M;$$

$$\max_{|s|\le r}\left|g(s)-\log\zeta_M\left(s+\frac{3}{4},\overline{\theta_0}\right)\right|<\varepsilon,$$

where

$$\overline{\theta_0}=\left(\frac{1}{4},\frac{2}{4},\frac{3}{4},\dots\right)$$

and

$$\log\zeta_M(s,\overline{\theta})=-\sum_{p\in M}\log\left(1-\frac{e^{-2\pi i\theta_p}}{p^s}\right).$$

The proof of this lemma can be found in [23](see Lemma 1 in Chapter 7).

Now we define the class $\mathcal{E}$ to consist of functions

$$F(s)=\prod_p R_p(p^{-s}),\quad \sigma>1,$$

where

$$R_p(z)=1+\sum_{m=1}^{\infty}a(p^m)z^m$$

are rational functions, analytic and non-vanishing on the disk $|z|<1$, which satisfy the following conditions:

1. (Ramanujan conjecture) $\forall_{\varepsilon>0}\ a(p^m)\ll_\varepsilon p^{\varepsilon m}$ uniformly in p.

2. F has meromorphic continuation to the half-plane $\sigma>1/2$. It can have at most a finite number of poles and all of them lie on the straight line $\sigma=1$.

3. F is a function of finite order, which means that

$$F(s) \ll_\sigma |t|^{A(\sigma)}, \quad \text{for } \sigma > 1, |t| \to \infty.$$

4. For any fixed $1/2 < \sigma < 1$, the square mean-value

$$\frac{1}{T}\int_{-T}^{T} |F(\sigma + it)|^2 dt$$

is bounded as $T \to \infty$.

Remark 4.3.4. The notation $A \ll_y B$ means that $A \ll B$ holds for fixed y.
Next lemma is Lemma 3.1 in [44].

Lemma 4.3.5. *Let G be an admisible domain such that*

$$\overline{G} \subset \left\{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1 \right\} \text{ and } \{F_1, \dots, F_n\} \subset \mathcal{E}$$

be any acceptable set. Moreover, let $(\alpha_i)_{1\le i\le m}$ be real numbers linearly independent over $\mathbb{Q}$, $(\theta_i)_{1\le i\le m}$ any real numbers, and $f_1, \dots, f_n$ functions which are analytic and non-vanishing on $\overline{G}$. Then, for every $\varepsilon > 0$ and any set $G_0 \subset \overline{G_0} \subset G$, there exist a finite set $B = B(\alpha_1, \dots, \alpha_m)$ of primes and a sequence $({\theta_p}^*)_{p\in B}$ of real numbers such that the set of positive real numbers τ satisfying the inequalities

$$\max_{1\le j\le n} \max_{s\in\overline{G_0}} \left| F_j(s + i\tau)_{|B} - f_j(s) \prod_{p\in B} {R_p}^{-1}(p^{-s} e(-{\theta_p}^*)) \right| < \varepsilon,$$

$$\max_{p\in B} \left\| \tau \frac{\log p}{2\pi} - {\theta_p}^* \right\| < \varepsilon,$$

$$\max_{1\le i\le m} \left\| \tau\alpha_i - \theta_i \right\| < \varepsilon$$

has a positive lower density.

The other auxiliary lemma which will be useful in the proof of Lemma 4.3.1 is as follows.

Lemma 4.3.6. *Let $F \in \mathcal{E}$ be an acceptable function, G any admissible set such that $\overline{G} \subset \mathcal{D}$ and functions f, g be analytic and non-vanishing on $\overline{G}$. Moreover, suppose that α, β are real numbers linearly independent over $\mathbb{Q}$ and the set A has the same meaning as in Lemma 4.2.1.*
Then, for every $\varepsilon > 0$ and any arbitrary set $G_0 \subset \overline{G_0} \subset G$, there exist finite sets $A_1 \subset \mathbb{P} \setminus A$, $A_2 \subset \mathbb{P}$ and real numbers ${\theta_p}^{(1)}$, $p \in A_1$, ${\theta_p}^{(2)}$, $p \in A_2$ such that the

set of real numbers τ satisfying the following inequalities

$$\max_{s\in\overline{G_0}}\left|F(s+i\alpha\tau)_{|(A\cup A_1)}-f(s)\prod_{p\in A}{R_p}^{-1}(p^{-s})\prod_{p\in A_1}{R_p}^{-1}(p^{-s}e(-{\theta_p}^{(1)}))\right|<\varepsilon,$$

$$\max_{s\in\overline{G_0}}\left|F(s+i\beta\tau)_{|A_2}-g(s)\prod_{p\in A_2}{R_p}^{-1}(p^{-s}e(-{\theta_p}^{(2)}))\right|<\varepsilon,$$

$$\max_{p\in A}\left\|\alpha\tau\frac{\log p}{2\pi}\right\|<\varepsilon,$$

$$\max_{p\in A_1}\left\|\alpha\tau\frac{\log p}{2\pi}-{\theta_p}^{(1)}\right\|<\varepsilon,$$

$$\max_{p\in A_2}\left\|\beta\tau\frac{\log p}{2\pi}-{\theta_p}^{(2)}\right\|<\varepsilon$$

has a positive lower density.

The proof of this lemma can be found in [43].

4.4 The proof of Lemma 4.3.1

In this section we prove main Lemma 4.3.1

Proof. We closely follow the proof of Lemma 4.3.3, Lemma 4.3.5, and Lemma 4.3.6.

At the beginning let us assume that $\varphi_1,\dots,\varphi_m$ is a basis of the vector space

$$\left\{\alpha\frac{\log p}{2\pi}\right\}_{p\in(A\cup B)}\cup\left\{\beta\frac{\log p}{2\pi}\right\}_{p\in B}.$$

Then there exists an integer N_1 such that every number $\alpha\frac{\log p}{2\pi}$ for $p\in A\cup B$ and $\beta\frac{\log p}{2\pi}$ for $p\in B$ can be expressed as a linear combination of φ_i/N_1 with integer coefficients.

Therefore,

$$\max_{1\le i\le m}||\tau\varphi_i/N_1||<\varepsilon$$

implies

$$\max_{\gamma\in\{\alpha,\beta\}}\max_{p\in B}\left\|\gamma\tau\frac{\log p}{2\pi}\right\|\ll\varepsilon,$$

$$\max_{p\in A}\left\|\alpha\tau\frac{\log p}{2\pi}\right\|\ll\varepsilon.$$

Now applying Lemma 4.2.2 for $\theta_i = 0$ $(1 \le i \le m)$, $\alpha_i = \varphi_i / N_1$ and

$$\{a_n\} = \left\{\alpha \frac{\log p}{2\pi}\right\}_{p \in \mathbb{P} \setminus (A \cup B)} \cup \left\{\beta \frac{\log p}{2\pi}\right\}_{p \in \mathbb{P} \setminus B}$$

we can choose sets

$$A_1 \subset \mathbb{P} \setminus (A \cup B), A_2 \subset \mathbb{P} \setminus B, J \subset \{1, 2, \dots, m\},$$

real numbers

$$\theta_p^{(1)}, p \in A_1, \theta_p^{(2)}, p \in A_2$$

and positive integer N such that

$$\max_{j \notin J} \left\| N \frac{\tau \varphi_j}{N_1} \right\| < \varepsilon$$

whenever the following inequalities hold:

$$\max_{j \in J} \left\| \frac{\tau \varphi_j}{N_1} \right\| < \frac{\varepsilon}{N}, \tag{4.4}$$

$$\max_{p \in A_1} \left\| \tau \frac{\alpha \log p}{2\pi} - \frac{\theta_p^{(1)}}{N} \right\| < \frac{\varepsilon}{N}, \qquad \max_{p \in A_2} \left\| \tau \frac{\beta \log p}{2\pi} - \frac{\theta_p^{(2)}}{N} \right\| < \frac{\varepsilon}{N}. \tag{4.5}$$

Let

$$\widetilde{f}_j(s) = f_j(s) \prod_{p \in A \cup B} \left(1 - \frac{\chi_j(p)}{p^s}\right) \prod_{p \in A_1} \left(1 - \frac{\chi_j(p) e(-\theta_p^{(1)})}{p^s}\right),$$

$$\widetilde{g}_j(s) = g_j(s) \prod_{p \in B} \left(1 - \frac{\chi_j(p)}{p^s}\right) \prod_{p \in A_2} \left(1 - \frac{\chi_j(p) e(-\theta_p^{(2)})}{p^s}\right).$$

Fix $\varepsilon > 0$. Then Lemma 4.2.4 yields that there exist sequences

$$\Theta_k = (\theta_p^{(k)}),\ \Lambda_k = (\lambda_p^{(k)}) \in \Omega$$

and an integer k_0 such that for each $k \ge k_0$

$$\max_{s \in \overline{G}} \left| L_{M_k^{(1)}}(s, \chi_j, \Theta_k) - \tilde{f}_j(s) \right| < \frac{\varepsilon}{2},$$

$$\max_{s \in \overline{G}} \left| L_{M_k^{(2)}}(s, \chi_j, \Lambda_k) - \tilde{g}_j(s) \right| < \frac{\varepsilon}{2},$$

where $M_k^{(1)}$ is a finite set of primes $p \notin A \cup A_1 \cup B$ and $M_k^{(2)}$ is a finite set of primes $p \notin A_2 \cup B$.

Let k denote a generic integer greater or equal to k_0. Then, by continuity, if

$$\max_{p \in M_k^{(1)}} \left\| N\tau \frac{\alpha \log p}{2\pi} - \theta_p^{(k)} \right\| < \delta, \tag{4.6}$$

$$\max_{p \in M_k^{(2)}} \left\| N\tau \frac{\beta \log p}{2\pi} - \lambda_p^{(k)} \right\| < \delta \tag{4.7}$$

for sufficiently small $\delta > 0$, then we have

$$\max_{s \in \overline{G}} \left| L_{M_k^{(1)}}(s + iN\alpha\tau, \chi_j, 0) - \widetilde{f}_j(s) \right| < \varepsilon, \tag{4.8}$$

$$\max_{s \in \overline{G}} \left| L_{M_k^{(2)}}(s + iN\beta\tau, \chi_j, 0) - \widetilde{g}_j(s) \right| < \varepsilon. \tag{4.9}$$

Now, by the choice of numbers φ_j and the sets J, A_1, A_2 and the first part of Lemma 4.2.2, one can apply Lemma 4.2.3 to obtain that the set $\mathcal{A}$ of positive numbers τ satisfying inequalities (4.4)–(4.7) has a positive density. Notice that for these τ we have

$$\max_{p \in (A \cup B)} \left\| N\tau \frac{\alpha \log p}{2\pi} \right\| \ll \varepsilon, \qquad \max_{p \in B} \left\| N\tau \frac{\beta \log p}{2\pi} \right\| \ll \varepsilon. \tag{4.10}$$

Now let us consider $I = I_\alpha + I_\beta$, where

$$I_\alpha = \frac{1}{T} \int\limits_{\mathcal{A}_T} \left(\iint\limits_G \left| L(s + iN\alpha\tau, \chi_j)_{|(A \cup A_1 \cup B)} - L_{M_k^{(1)}}(s + iN\alpha\tau, \chi_j, 0) \right|^2 db \right) d\tau,$$

$$I_\beta = \frac{1}{T} \int\limits_{\mathcal{A}_T} \left(\iint\limits_G \left| L(s + iN\beta\tau, \chi_j)_{|(A_2 \cup B)} - L_{M_k^{(2)}}(s + iN\beta\tau, \chi_j, 0) \right|^2 db \right) d\tau,$$

with $db = d\sigma dt$ and $\mathcal{A}_T = \mathcal{A} \cap [1, T]$.

Arguing analogously as in [23] or [44], we prove that $I \ll \varepsilon^2$. The modifications needed are easy and can be left to the reader.

Therefore, there exists a set $Y \subset \mathcal{A}_T$ such that $\mu(Y) \gg T$ and for all $\tau \in Y$ the following inequalities hold:

$$\max_{s \in \overline{G_0}} \left| L(s + iN\alpha\tau, \chi_j)_{|(A \cup A_1 \cup B)} - L_{M_k^{(1)}}(s + iN\alpha\tau, \chi_j, 0) \right| \ll \varepsilon,$$

$$\max_{s \in \overline{G_0}} \left| L(s + iN\beta\tau, \chi_j)_{|(A_2 \cup B)} - L_{M_k^{(2)}}(s + iN\beta\tau, \chi_j, 0) \right| \ll \varepsilon,$$

where G_0 is an arbitrary set such that $G_0 \subset \overline{G_0} \subset G$. Hence, taking $\tau' = N\tau$ and recalling (4.8),(4.9), and the definition of $\mathcal{A}$ complete the proof. $\square$

4.5 Auxiliary Theorem 4.5.1

Using the previous Lemma 4.3.1 we prove the following theorem.

Theorem 4.5.1. *Let $\mathcal{K} \subset \mathcal{D}$ be any compact set with connected complement, $\chi_1, \dots, \chi_n$ be pairwise non-equivalent Dirichlet characters, and $f_j, g_j, (j = 1, \dots, n)$ be functions which are non-vanishing and continuous on $\mathcal{K}$ and analytic in the interior. Moreover, let α, β be real numbers linearly independent over $\mathbb{Q}$ and B be a finite set of prime numbers.*

Then, for every $\varepsilon > 0$, the set of real numbers τ satisfying

$$\max_{1 \le j \le n} \max_{s \in \mathcal{K}} |L(s + i\alpha\tau, \chi_j) - f_j(s)| < \varepsilon, \tag{4.11}$$

$$\max_{1 \le j \le n} \max_{s \in \mathcal{K}} |L(s + i\beta\tau, \chi_j) - g_j(s)| < \varepsilon, \tag{4.12}$$

$$\max_{p \in B} \left\| \tau \frac{(\alpha - \beta) \log p}{2\pi} \right\| < \varepsilon \tag{4.13}$$

has a positive lower density.

Particularly, taking $f_j = g_j$ yields that the set of $\tau \in \mathbb{R}$ satisfying

$$\max_{1 \le j \le n} \max_{s \in \mathcal{K}} |L(s + i\alpha\tau, \chi_j) - L(s + i\beta\tau, \chi_j)| < \varepsilon,$$

$$\max_{p \in B} \left\| \tau \frac{(\alpha - \beta) \log p}{2\pi} \right\| < \varepsilon$$

has a positive lower density.

Remark 4.5.2. It should be noted that the above theorem can be easily generalized to more general L-functions which satisfy some natural analytic and arithmetic conditions, and an analog of Lemma 4.2.4. This wide class of L-functions was introduced and studied by Ł. Pańkowski in [44].

In the proof of Theorem 4.5.1 we make use of the following famous Mergelyan theorem [11].

Lemma 4.5.3. *Let $\mathcal{K} \subset \mathbb{C}$ be a compact set with connected complement and $f : \mathcal{K} \to \mathbb{C}$ any function continuous on $\mathcal{K}$ and analytic in the interior of $\mathcal{K}$. Then, for every $\varepsilon > 0$, there exists a polynomial P such that*

$$\max_{s \in \mathcal{K}} |f(s) - P(s)| < \varepsilon.$$

Proof of Theorem 4.5.1. By the Mergelyan theorem, we can assume that $\mathcal{K} = \overline{G}$ for some admissible set G and $f_j, g_j, (j = 1, \dots, n)$ are analytic and non-vanishing on some simply connected set G_1 such that $\overline{G} \subset G_1 \subset \overline{G_1} \subset D$.

By continuity, for any $\varepsilon > 0$, we find $\delta = \delta(\varepsilon)$ such that

$$\max_{p \in A \cup A_1 \cup A_2 \cup B} \max_{s \in \overline{G}} \left| \left(1 - \frac{\chi_j(p) e\big(-\omega_p^{(1)}\big)}{p^s}\right) - \left(1 - \frac{\chi_j(p) e\big(-\omega_p^{(2)}\big)}{p^s}\right) \right| < \varepsilon, \quad (4.14)$$

for all $j = 1, \dots, n$ whenever

$$\big\|\omega_p^{(1)} - \omega_p^{(2)}\big\| \le \delta \text{ for } p \in A \cup A_1 \cup A_2 \cup B.$$

Now, using Lemma 4.3.1, we see that the set of τ satisfying

$$\begin{aligned} &\max_{1 \le j \le n} \max_{s \in \overline{G_0}} \Big| L(s + i\alpha\tau, \chi_j)_{|(A \cup A_1 \cup B)} \\ &- f_j(s) \prod_{p \in A \cup B} \left(1 - \frac{\chi_j(p)}{p^s}\right) \prod_{p \in A_1} \left(1 - \frac{\chi_j(p) e(-\theta_p^{(1)})}{p^s}\right) \Big| < \varepsilon, \end{aligned}$$

$$\begin{aligned} &\max_{1 \le j \le n} \max_{s \in \overline{G_0}} \Big| L(s + i\beta\tau, \chi_j)_{|(A_2 \cup B)} \\ &- g_j(s) \prod_{p \in B} \left(1 - \frac{\chi_j(p)}{p^s}\right) \prod_{p \in A_2} \left(1 - \frac{\chi_j(p) e(-\theta_p^{(2)})}{p^s}\right) \Big| < \varepsilon, \end{aligned}$$

$$\max_{p \in A} \left\| \alpha\tau \frac{\log p}{2\pi} \right\| < \delta,$$

$$\max_{\gamma \in \{\alpha, \beta\}} \max_{p \in B} \left\| \gamma\tau \frac{\log p}{2\pi} \right\| < \delta,$$

$$\max_{p \in A_1} \left\| \alpha\tau \frac{\log p}{2\pi} - \theta_p^{(1)} \right\| < \delta,$$

$$\max_{p \in A_2} \left\| \beta\tau \frac{\log p}{2\pi} - \theta_p^{(2)} \right\| < \min(\delta, \varepsilon)$$

has a positive lower density.

Therefore, by (4.14), we obtain that, for $s \in \overline{G}$, we have

$$\begin{aligned} &\Big| f_j(s) \prod_{p \in A \cup B} \left(1 - \frac{\chi_j(p)}{p^s}\right) \prod_{p \in A_1} \left(1 - \frac{\chi_j(p) e(-\theta_p^{(1)})}{p^s}\right) \\ &- f_j(s) \prod_{p \in A \cup A_1 \cup B} \left(1 - \frac{\chi_j(p)}{p^{s + i\alpha\tau}}\right) \Big| \ll \varepsilon. \end{aligned}$$

Consequently, we have

$$\max_{s\in\overline{G}}\left|L(s+i\alpha\tau,\chi_j)_{|(A\cup A_1\cup B)}-f_j(s)\prod_{p\in A\cup A_1\cup B}\left(1-\frac{\chi_j(p)}{p^{s+i\alpha\tau}}\right)\right|\ll\varepsilon.$$

Multiplying the last inequality by

$$\prod_{p\in A\cup A_1\cup B}\left|\left(1-\frac{\chi_j(p)}{p^{s+i\alpha\tau}}\right)^{-1}\right|$$

and noticing that this factor is $O(1)$ gives (4.11).

Arguing analogously, one can prove (4.12). Moreover, (4.13) follows immediately from

$$\max_{\gamma\in\{\alpha,\beta\}}\max_{p\in B}\left\|\gamma\tau\frac{\log p}{2\pi}\right\|<\varepsilon.$$ □

Remark 4.5.4. It should be noted that the condition that $\mathcal{K}$ has a connected complement is needed only to apply the Mergelyan theorem 4.5.3. Thus, the second part of Theorem 4.5.1 can be proved for any compact set $\mathcal{K}$ by following all steps of the above proof for $f_j=g_j\equiv 1$.

4.6 The proof of Theorem 4.1.1

In this section we present the proof of the main theorem in this chapter.

Proof. We know that

$$\zeta\left(s+i\alpha\tau,\frac{a}{b}\right)=\frac{b^{s+i\alpha\tau}}{\varphi(b)}\sum_{\chi\mod b}\overline{\chi(a)}L(s+i\alpha\tau,\chi),$$

$$\zeta\left(s+i\beta\tau,\frac{a}{b}\right)=\frac{b^{s+i\beta\tau}}{\varphi(b)}\sum_{\chi\mod b}\overline{\chi(a)}L(s+i\beta\tau,\chi).$$

Then using Theorem 4.5.1 for all Dirichlet characters χ mod b and the set

$$B=\{p\in\mathbb{P}:p|b\}$$

yields

$$\max_{\chi\mod b}\max_{s\in\mathcal{K}}|L(s+i\alpha\tau,\chi)-L(s+i\beta\tau,\chi)|<\varepsilon \tag{4.15}$$

and

$$\max_{p|b}\left\|\tau\frac{(\alpha-\beta)\log p}{2\pi}\right\|<\varepsilon.$$

The last inequality implies that

$$\left\| \tau \frac{(\alpha-\beta)\log b}{2\pi} \right\| \ll \varepsilon. \tag{4.16}$$

Hence

$$|b^{s+i\alpha\tau} - b^{s+i\beta\tau}| = |b^{\sigma}||b^{i(\alpha-\beta)\tau} - 1| \ll |b^{i(\alpha-\beta)\tau} - 1| \ll \varepsilon.$$

Now (4.15), (4.16) and the fact that $b^s \ll 1$ for $s \in \mathcal{K}$ and $L(s+i\beta\tau, \chi)$ is bounded, provided $f_j = g_j \equiv 1$ in Theorem 4.5.1, yield

$$\max_{s\in\mathcal{K}} \left| \zeta\left(s+i\alpha\tau, \frac{a}{b}\right) - \zeta\left(s+i\beta\tau, \frac{a}{b}\right) \right| \ll \varepsilon,$$

and the theorem follows. □

Chapter 5

Self-approximation for periodic Hurwitz zeta-functions

5.1 Main Theorems 5.1.1 and 5.1.2

Let $\mathfrak{A} = \{c_m : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with the smallest period $k \in \mathbb{N}$.
For $\sigma > 1$, the periodic Hurwitz zeta-function is defined by

$$\zeta(s, \omega; \mathfrak{A}) = \sum_{m=0}^{\infty} \frac{c_m}{(m+\omega)^s}.$$

In this chapter, we prove two theorems which are generalizations of Theorem 3.4.1 and Theorem 4.1.1. The same notations as in previous chapters will be used.

Theorem 5.1.1. *Let $\mathfrak{A} = \{c_m : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with the smallest period $k \in \mathbb{N}$. Let $\omega = \frac{a}{b}$, $0 < a < b$ and $gcd(a,b) = 1$. Moreover, suppose that α, β are real numbers linearly independent over $\mathbb{Q}$ and $\mathcal{K}$ is any compact subset of the strip $1/2 < \sigma < 1$. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0,T] : \max_{s \in \mathcal{K}} \left| \zeta\left(s + i\alpha\tau, \frac{a}{b}; \mathfrak{A}\right) - \zeta\left(s + i\beta\tau, \frac{a}{b}; \mathfrak{A}\right) \right| < \varepsilon \right\} > 0.$$

In Theorem 5.1.1, we consider the case when the parameter ω is a rational number. In the next theorem, we consider the case when the parameter ω is a transcendental number.

Theorem 5.1.2. *Let $\mathfrak{A} = \{c_m : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with the smallest period $k \in \mathbb{N}$. Let ω be a transcendental number from the interval $(0, 1]$. Moreover, suppose that $\alpha, \beta \in \mathbb{R}$ are such that $A(\alpha, \beta; \omega)$ is linearly independent over $\mathbb{Q}$ and $\mathcal{K}$ is any compact subset of the strip $1/2 < \sigma < 1$.*

Then, for any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0,T] : \max_{s\in\mathcal{K}} \left| \zeta\left(s+i\alpha\tau, \omega; \mathfrak{A}\right) - \zeta\left(s+i\beta\tau, \omega; \mathfrak{A}\right) \right| < \varepsilon, \right.$$
$$\left. \left|\left| \frac{(\alpha-\beta)\tau \log k}{2\pi} \right|\right| < \varepsilon \right\} > 0.$$

In the next section, we prove Theorem 5.1.1. Section 5.3 is devoted to the proof of Theorem 5.1.2.

5.2 Proof of Theorem 5.1.1

Theorem 5.1.1 will be derived from the following proposition.

Proposition 5.2.1. *Let $k, n \in \mathbb{N}$ and $\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}$ be rational numbers satisfying $0 < a_j < b_j$ and $gcd(a_j, b_j) = 1$ for $j = 1, 2, \dots, n$. Moreover, suppose that α, β are real numbers linearly independent over $\mathbb{Q}$ and $\mathcal{K}$ is any compact subset of the strip $1/2 < \sigma < 1$. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0,T] : \max_{s\in\mathcal{K}} \max_{1\le j\le n} \left| \zeta\left(s+i\alpha\tau, \frac{a_j}{b_j}\right) - \zeta\left(s+i\beta\tau, \frac{a_j}{b_j}\right) \right| < \varepsilon, \right.$$
$$\left. \max_{p|k} \left|\left| \frac{1}{2\pi}\tau \log p \right|\right| < \varepsilon \right\} > 0. \tag{5.1}$$

Note that the inequality

$$\max_{p|k} \left|\left| \frac{1}{2\pi}\tau \log p \right|\right| < \varepsilon$$

implies that

$$\max_{s\in\mathcal{K}} |k^{s+i\tau} - k^s| \ll \varepsilon.$$

Proof. Let us consider the set of the functions $\left\{ \zeta(s, \frac{a_1}{b_1}), \zeta(s, \frac{a_2}{b_2}), \dots, \zeta(s, \frac{a_n}{b_n}) \right\}$.

Since $(a_j, b_j) = 1$ $(j = 1, \dots n)$, we have

$$\zeta\left(s, \frac{a_j}{b_j}\right) = \frac{b_j^s}{\varphi(b_j)} \sum_{\chi^{(j)} \bmod b_j} \overline{\chi^{(j)}(a_j)} L\left(s, \chi^{(j)}\right) = \frac{b_j^s}{\varphi(b_j)} \sum_{k=1}^{\varphi(b_j)} \overline{\chi_k^{(j)}(a_j)} L\left(s, \chi_k^{(j)}\right).$$

Thus

$$\zeta\left(s, \frac{a}{b}, \mathfrak{A}\right) = \frac{1}{k^s} \sum_{l=0}^{k-1} c_l \frac{b_l^s}{\varphi(b_l)} \sum_{\chi^{(l)} \bmod b_l} \overline{\chi^{(l)}(a_l)} L\left(s, \chi^{(l)}\right),$$

where $\frac{a+bl}{bk} = \frac{a_l}{b_l}$, $(a_l, b_l) = 1$ for all $0 \le l \le k-1$.

Two characters, $\chi_1 \bmod k_1$, $\chi_2 \bmod k_2$, are equivalent if they are induced by the same primitive character $\chi^* \bmod k$ with $k|k_1$ and $k|k_2$. Then, for $j = 1, 2$, we have

$$L(s, \chi_j) = L(s, \chi^*) \prod_{p|k_j} \left(1 - \frac{\chi^*(p)}{p^s}\right).$$

Now let us assume that $\chi_k^{(j)}$ is induced by a primitive character $\chi^{(j)*}$. Let us observe that every two elements from the set

$$\left\{\chi_1^{(1)*}, \chi_2^{(1)*}, \dots, \chi_{\varphi(b_1)}^{(1)*}, \dots, \chi_1^{(n)*}, \chi_2^{(n)*}, \dots, \chi_{\varphi(b_n)}^{(n)*}\right\}$$

are non-equivalent either equal.
Let $\chi_1, \dots \chi_N$ denote all distinct characters in the set

$$\left\{\chi_1^{(1)*}, \chi_2^{(1)*}, \dots, \chi_{\varphi(b_1)}^{(1)*}, \dots, \chi_1^{(n)*}, \chi_2^{(n)*}, \dots, \chi_{\varphi(b_n)}^{(n)*}\right\}.$$

Moreover, put

$$P\left(s, \chi^{(j)}\right) = \begin{cases} 1 & \text{if } \chi^{(j)} \text{ is primitive} \\ \prod_{p|q} \left(1 - \frac{\chi^{(j)*}(p)}{p^s}\right) & \text{if } \chi^{(j)} \text{ is imprimitive character mod } q. \end{cases}$$

Let us observe that, for any imprimitive character $\chi^{(j)} \bmod q$, we have

$$|P(s + i\tau, \chi^{(j)}) - P(s, \chi^{(j)})| \ll \varepsilon,$$

provided

$$\max_{p|q} \left\| \frac{1}{2\pi} \tau \log p \right\| \ll \varepsilon.$$

Therefore

$$\zeta\left(s, \frac{a_j}{b_j}\right) = \frac{b_j^s}{\varphi(b_j)} \sum_{k=1}^{\varphi(b_j)} \overline{\chi_k^{(j)}(a_j)} P(s, \chi_k^{(j)}) L(s, \chi_k^{(j)}).$$

We see that, for any $\varepsilon > 0$, there are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\left|\zeta\left(s + i\tau, \frac{a_j}{b_j}\right) - \zeta\left(s, \frac{a_j}{b_j}\right)\right| < \varepsilon$$

for all $j = 1, \dots, n$, if

$$\begin{aligned} |L(s + i\tau, \chi_r) - L(s, \chi_r)| &< \varepsilon_1 \quad \text{for all } r = 1, \dots, N, \\ \left|P\left(s + i\tau, \chi_r^{(j)}\right) - P\left(s, \chi_r^{(j)}\right)\right| &< \varepsilon_2 \quad \text{for all } j = 1, \dots, n,\ r = 1, \dots, \varphi(b_j). \end{aligned} \tag{5.2}$$

The above inequalities (5.2) are implied by Theorem 4.5.1. This proves Proposition 5.2.1. $\square$

Proof of Theorem 5.1.1. Let $\omega = \frac{a}{b}$, $0 < a < b$ and $gcd(a, b) = 1$. Then, for $\sigma > 1$, we have

$$\zeta\Big(s, \frac{a}{b}; \mathfrak{A}\Big) = \sum_{l=0}^{k-1} \sum_{m=0}^{\infty} \frac{c_l}{\Big(mk + l + \frac{a}{b}\Big)^s} = \frac{1}{k^s} \sum_{l=0}^{k-1} c_l \sum_{m=0}^{\infty} \frac{1}{\Big(m + \Big(l + \frac{a}{bk}\Big)\Big)^s}$$
$$= \frac{1}{k^s} \sum_{l=0}^{k-1} c_l \zeta\left(s, \frac{l + \frac{a}{b}}{k}\right) = \frac{1}{k^s} \sum_{l=0}^{k-1} c_l \zeta\left(s, \frac{lb + a}{bk}\right).$$

Obviously, for all l with $0 \leq l \leq k - 1$, we can find a_l, b_l such that $(a_l, b_l) = 1$ and $\frac{lb+a}{bk} = \frac{a_l}{b_l}$. Hence

$$\zeta\left(s, \frac{a}{b}, \mathfrak{A}\right) = \frac{1}{k^s} \sum_{l=0}^{k-1} c_l \zeta\left(s, \frac{a_l}{b_l}\right).$$

Now we have

$$\begin{aligned} &\max_{s \in \mathcal{K}} |\zeta(s + i\alpha\tau, \omega; \mathfrak{A}) - \zeta(s + i\beta\tau, \omega; \mathfrak{A})| \\ &= \max_{s \in \mathcal{K}} \left| \frac{1}{k^{s+i\alpha\tau}} \sum_{l=0}^{k-1} c_l \zeta\left(s + i\alpha\tau, \frac{a_l}{b_l}\right) - \frac{1}{k^{s+i\beta\tau}} \sum_{l=0}^{k-1} c_l \zeta\left(s + i\beta\tau, \frac{a_l}{b_l}\right) \right| \\ &\leq \max_{s \in \mathcal{K}} \max_{0 \leq l \leq k-1} |kc_l| \left| \frac{1}{k^{s+i\alpha\tau}} \zeta\left(s + i\alpha\tau, \frac{a_l}{b_l}\right) - \frac{1}{k^{s+i\beta\tau}} \zeta\left(s + i\beta\tau, \frac{a_l}{b_l}\right) \right|. \end{aligned} \tag{5.3}$$

Note that $|kc_l| \ll 1$.
In view of (5.3), it is easy to see that Theorem 5.1.1 follows from Proposition 5.2.1.

$\square$

5.3 Proof of Theorem 5.1.2

Proof of Theorem 5.1.2. Let α be a real number. By Proposition 3.4.3, we can find a real number β such that the set $A(\alpha, \beta; \omega)$ is linearly independent over $\mathbb{Q}$.

We have that

$$\max_{s \in \mathcal{K}} |\zeta(s + i\alpha\tau, \omega; \mathfrak{A}) - \zeta(s + i\beta\tau, \omega; \mathfrak{A})|$$

$$= \max_{s\in\mathcal{K}} \left| \frac{1}{k^{s+i\alpha\tau}} \sum_{l=0}^{k-1} c_l \zeta\left(s+i\alpha\tau,\omega_l\right) - \frac{1}{k^{s+i\beta\tau}} \sum_{l=0}^{k-1} c_l \zeta\left(s+i\beta\tau,\omega_l\right) \right|$$

$$\le \max_{s\in\mathcal{K}} \max_{0\le l\le k-1} |kc_l| \left| \frac{1}{k^{s+i\alpha\tau}} \zeta\left(s+i\alpha\tau,\omega_l\right) - \frac{1}{k^{s+i\beta\tau}} \zeta\left(s+i\beta\tau,\omega_l\right) \right|. \tag{5.4}$$

Note that $|kc_l| \ll 1$.
Inequality

$$\left\| \tau \frac{(\alpha-\beta)\log k}{2\pi} \right\| < \varepsilon$$

implies that

$$|k^{s+i\alpha\tau} - k^{s+i\beta\tau}| = |k^{\sigma}||k^{i(\alpha-\beta)\tau} - 1| \ll |k^{i(\alpha-\beta)\tau} - 1| \ll \varepsilon.$$

This means that

$$\frac{1}{k^{s+i\alpha\tau}} \text{ is near } \frac{1}{k^{s+i\beta\tau}}.$$

Now we consider linear independence of numbers $\log(n+\omega_l)$ $(n \in \mathbb{N}_0)$ and $\log k$ over $\mathbb{Q}$, where $\omega_l = \frac{l+\omega}{k}$ and $l = 0, \dots, k-1$.

Assume that there exists a finite sequence of rational numbers

$$a_{ln}, l = 0, \dots, k-1, n = 0, 1, 2, \dots, N \text{ and } d$$

such that not all of these numbers are equal to 0 and

$$\sum_{l=0}^{k-1} \sum_{n=0}^{N} a_{ln} \log(n+\omega_l) + d\log k \tag{5.5}$$

$$= \sum_{l=0}^{k-1} \sum_{n=0}^{N} a_{ln} (\log(nk+l+\omega) - \log k) + d\log k = 0.$$

Then

$$\sum_{l=0}^{k-1} \sum_{n=0}^{N} a_{ln} \log(nk+l+\omega) = \log k^{\gamma},$$

where

$$\gamma = \sum_{l=0}^{k-1} \sum_{n=0}^{N} a_{ln} - d$$

and

$$\prod_{l=0}^{k-1}\prod_{n=0}^{N}(nk+l+\omega)^{a_{ln}}=k^{\gamma}. \tag{5.6}$$

Numbers a_{ln}, d and γ are rationals. Therefore, it is not difficult to see that we can write (5.6) in the form $P(\omega)=0$, where $P(\omega)$ is a polynomial. Then ω is a root of this polynomial. But ω is a transcendental number and we obtain a contradiction. This gives that numbers $\log(n+\omega_l)$ and $\log k$ are linearly independent over $\mathbb{Q}$.

By the linear independence of numbers $\log(n+\omega_l)$ and $\log k$ over $\mathbb{Q}$, and by Theorem 3.4.1 (for $m=2$) we obtain:

$$\max_{s\in\mathcal{K}}\max_{0\le l\le k-1}\left|\zeta\left(s+i\alpha\tau,\omega_l\right)-\zeta\left(s+i\beta\tau,\omega_l\right)\right|\ll\varepsilon$$

and Theorem 5.1.2 follows. $\square$

Conclusions

The results of our thesis demonstrate that:

- Hurwitz zeta-functions $\zeta(s,\omega)$ have the self-approximation property if ω is a transcendental or a rational number.
- Periodic Hurwitz zeta-functions $\zeta(s,\omega;\mathfrak{A})$ have the self-approximation property if ω is a transcendental or a rational number.
- The case of algebraic irrational ω is the most difficult case in this context (there is no approach to treat these questions in this case).

Santrauka

Šioje disertacijoje nagrinėjamos Hurvico (Hurwitz) dzeta funkcijų ir periodinių Hurvico dzeta funkcijų saviaproksimacijos.

Tegu $s = \sigma + it$ yra kompleksinis kintamasis, o ω – realusis skaičius iš intervalo (0,1]. Kai $\sigma > 1$, tai Hurvico dzeta funkcija apibrėžiama lygybe

$$\zeta(s, \omega) = \sum_{n=0}^{\infty} \frac{1}{(n+\omega)^s}.$$

Tarkime, kad $\mathfrak{A} = \{c_m : m \in \mathbb{N}_0\}, \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ yra periodinė kompleksinių skaičių seka, kurios mažiausias teigiamas periodas $k \in \mathbb{N}$. Kai $\sigma > 1$, tai periodinė Hurvico dzeta funkcija apibrėžiama eilute

$$\zeta(s, \omega; \mathfrak{A}) = \sum_{m=0}^{\infty} \frac{c_m}{(m+\omega)^s}.$$

- Mes parodėme, kad galioja saviaproksimacijos savybė Hurvico dzeta funkcijai, kai ω yra trancendentusis skaičius iš intervalo (0,1].

  Tarkime, kad $l \leq m$ ir $d_1, \dots, d_l \in \mathbb{R}$ tokie, kad aibė

  $$A(d_1, d_2, \dots, d_l; \omega) = \{d_j \log(n+\omega) : j = 1, \dots, l; n \in \mathbb{N}_0\}$$

  yra tiesiškai neprikalausoma virš $\mathbb{Q}$. Kai $m > l$, tegu $d_{l+1}, \dots, d_m \in \mathbb{R}$ tokie, kad kiekvienas d_k, $k = l+1, \dots, m$, yra skaičių $d_1, \dots, d_l$ tiesinė kombinacija virš $\mathbb{Q}$. Mes parodėme, kad yra 'daug' skaičių $\tau \in \mathbb{R}$, su kuriais reikšmės $\zeta(s + id_j\tau, \omega)$ ir $\zeta(s + id_k\tau, \omega)$ yra 'artimos', čia $1 \leq j, k \leq m$. Tai papildo R. Garunkščio gautą rezultatą Dirichlė L funkcijoms.

  Be to, parodėme, kad su kiekvienu $l > 0$ 'dauguma' realiųjų skaičių rinkinių $d_1, d_2, \dots, d_l, \omega$, kai $0 < \omega \leq 1$, yra tokie, kad aibė $A(d_1, d_2, \dots, d_l; \omega)$ yra tiesiškai nepriklausoma virš $\mathbb{Q}$.

- Mes taip pat parodėme, kad galioja saviaproksimacijos savybė Hurvico dzeta funkcijoms, kai parametras yra racionalusis skaičius.

Primename, kad kai $\omega = \frac{a}{b}$ yra racionalusis skaičius, (čia $0 < a < b$ ir $(a, b) = 1$), tai Hurvico dzeta funkciją galima išreikšti Dirichlė L funkcijų tiesine kombinacija

$$\zeta\left(s + i\tau, \frac{a}{b}\right) = \frac{b^{s+i\tau}}{\varphi(b)} \sum_{\chi \bmod b} \overline{\chi(a)} L(s + i\tau, \chi).$$

Taigi, pasirėmę paskutine lygybe, įrodėme, kad $\zeta(s+i\alpha\tau, \frac{a}{b})$ tolygiai aproksimuoja $\zeta(s + i\beta\tau, \frac{a}{b})$ su be galo daug realiųjų skaičių τ. Čia α, β yra realieji skaičiai, tiesiškai nepriklausomi virš $\mathbb{Q}$, o s yra iš kompaktinės aibės $\mathcal{K}$, kuri priklauso kritinei juostai $1/2 < \sigma < 1$. Šis rezultatas papildo Pankovskio rezultatą gautą Dirichlė L funkcijoms.

- Pasinaudoję ankstesniais mūsų gautais rezultatais, taip pat parodėme, kad saviaproksimacijos savybė galioja ir periodinėms Hurvico dzeta funkcijoms $\zeta(s, \omega; \mathfrak{A})$. Mes išnagrinėjome atvejus, kai ω yra racionalusis ir trancendentusis skaičiai.

Šioje disertacijoje gautiems rezultatams įrodyti buvo taikyti metodai, nagrinėti neseniai gautuose Garunkščio [12] ir Pankovskio [43], [44] darbuose. Taip pat taikyti kompleksinio kintamojo funkcijų elementai, mato teorija ir diofantiniai metodai.

Notations

$a, b, j, k, l, m, n, r, t$ – positive integer numbers.

$\mathbb{Q}$ – the set of all rational numbers.

$|s|$ – the absolute value of a complex number s.

$\mathcal{K}$ – the compact subset of the strip $1/2 < \sigma < 1$.

$\|x\|$ – the distance from $x \in \mathbb{R}$ to the nearest integer.

(a, b) or $\gcd(a, b)$ – the greatest common divisor of a and b.

$\mathfrak{A}$ – the periodic sequence of complex numbers with the smallest period $k \in \mathbb{N}$.

$\operatorname{meas}\{A\}$ – the Lebesgue measure of a measurable set A.

$\varphi(n)$ – the Euler totient function.

$\ll$ – means that $f(x) \ll g(x)$ if and only if there a positive number c and a real number x_0 such that $|f(x)| \leq c|g(x)|$, for all $x > x_0$.

$\ll_y$ – means that $A \ll B$ holds for fixed y.

ε, δ – arbitrarily small positive number.

χ – the Dirichlet character.

$\log x$ – the natural logarithm of x.

$\Gamma(s)$ – the Euler gamma-function defined by

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx, \text{ for } \sigma > 0,$$

and by analytic continuation elsewhere.

$\zeta(s)$ – the Riemann zeta-function defined by

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}, \text{ for } \sigma > 1,$$

and by analytic continuation elsewhere.

$\zeta(s,\omega)$ – the classical Hurwitz zeta-function defined by

$$\zeta(s,\omega) = \sum_{n=0}^\infty \frac{1}{(n+\omega)^s}, \text{ for } \sigma > 1,$$

and by analytic continuation elsewhere.

$\zeta(s,\omega;\mathfrak{A})$ – the periodic Hurwitz zeta-function defined by

$$\zeta(s,\omega;\mathfrak{A}) = \sum_{m=0}^\infty \frac{c_m}{(m+\omega)^s}, \text{ for } \sigma > 1,$$

and by analytic continuation elsewhere.

$L(s,\chi)$ – the Dirichlet L-function defined by

$$L(s,\chi) = \sum_{n=1}^\infty \frac{\chi(n)}{n^s}, \text{ for } \sigma > 1,$$

and by analytic continuation elsewhere.

$L(\lambda,\omega,s)$ – the Lerch zeta-function defined by

$$L(\lambda,\omega,s) = \sum_{m=0}^\infty \frac{e^{2\pi i\lambda m}}{(m+\omega)^s}, \text{ for } \sigma > 1,$$

and here $\omega, \lambda \in \mathbb{R}$, $0 < \omega \le 1$, are fixed parameters.

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