mathematics

Article

Discrete Approximation by a Dirichlet Series Connected to the Riemann Zeta-Function

Antanas Laurinčikas [1,†] and Darius Šiaučiūnas [2,*,†]

[1] Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko Str. 24, LT-03225 Vilnius, Lithuania; antanas.laurincikas@mif.vu.lt
[2] Regional Development Institute, Šiauliai Academy, Vilnius University, P. Višinskio Str. 25, LT-76351 Šiauliai, Lithuania
* Correspondence: darius.siauciunas@sa.vu.lt
† These authors contributed equally to this work.

Abstract: In the paper, a Dirichlet series $\zeta_{u_N}(s)$ whose shifts $\zeta_{u_N}(s+ikh)$, $k=0,1,\dots$, $h>0$, approximate analytic non-vanishing functions defined on the right-hand side of the critical strip is considered. This series is closely connected to the Riemann zeta-function. The sequence $u_N \to \infty$ and $u_N \ll N^2$ as $N \to \infty$.

Keywords: distribution function; Riemann zeta-function; Voronin universality theorem; weak convergence

MSC: 11M06; 11M41

Citation: Laurinčikas, A.; Šiaučiūnas, D. Discrete Approximation by a Dirichlet Series Connected to the Riemann Zeta-Function. *Mathematics* **2021**, *9*, 1073. https://doi.org/10.3390/math9101073

Academic Editor: Shanhe Wu

Received: 21 April 2021
Accepted: 7 May 2021
Published: 11 May 2021

1. Introduction

Denote by $s=\sigma+it$ a complex variable and by $\mathbb{P}$ the set of all prime numbers. The Riemann zeta-function $\zeta(s)$ is defined, for $\sigma>1$, by

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}=\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^s}\right)^{-1},$$

and has an analytic continuation to the whole complex plane, except for a simple pole at the point $s=1$ with residue 1.

By the Voronin theorem [1], the function $\zeta(s)$ is universal in the sense that its shifts $\zeta(s+i\tau)$, $\tau\in\mathbb{R}$, approximate uniformly on compact sets every analytic non-vanishing function defined on the strip $D=\{s\in\mathbb{C}:1/2<\sigma<1\}$. We recall that the original version of the Voronin theorem asserts that if $f(s)$ is a continuous non-vanishing function on the disc $|s|\leqslant r$, $0<r<1/4$, and analytic on $|s|<r$, then, for every $\varepsilon>0$, there exists $\tau=\tau(\varepsilon)$ such that

$$\max_{|s|\leqslant r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-f(s)\right|<\varepsilon.$$

Voronin considered a general case when τ takes arbitrary real values (continuous case). Reich introduced [2] a discrete version of the Voronin theorem with values τ from the set $\{kh:k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\}$ with arbitrary fixed $h>0$. By a different method, a discrete universality theorem for $\zeta(s)$ was also proved by Bagchi [3]. The arithmetic progression $\{kh:k\in\mathbb{N}_0\}$ is one of the simplest discrete sets, and its case is classical in the theory of universality. In view of the equality $\zeta(s+ikh)=\overline{\zeta(\overline{s}-ikh)}$, where $\overline{z}$ means the conjugate of z, the case $h<0$ does not give new results. Of course, in place of $\{kh:k\in\mathbb{N}_0\}$, more general sets $\{\varphi(k):k\in\mathbb{N}_0\}$ with a function satisfying some hypotheses can be considered. For example, in [4] the function $\varphi(k)=k^\alpha\log^\beta k$, $k\geqslant 2$, with certain $\alpha,\beta\in\mathbb{R}$ was used. Let $\mathcal{K}$ be the class of compact subsets of the strip D with connected complements, and $H_0(K)$

with $K \in \mathcal{K}$ be the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then the above-mentioned discrete universality of $\zeta(s)$ is contained in the following theorem.

Theorem 1 ([2])**.** *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\left\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon \right\} > 0.$$

Here $\#A$ denotes the cardinality of the set A, and N runs over the set $\mathbb{N}_0$.

Mauclaire observed [5] that a lower density in universality theorems can be replaced by a density with a certain exception for ε. Thus, the following statement is true.

Theorem 2. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $h > 0$, the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \#\left\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The main aim of this paper is to replace $\zeta(s)$ in Theorem 2 by a certain absolutely convergent for $\sigma > 1/2$ Dirichlet series depending on N.

2. Statement of the Main Theorem

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, and by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta. For the proof of universality theorems, we will apply the probabilistic Bagchi approach based on limit theorems for probabilistic measures in the space $H(D)$. Two types of $h > 0$ are considered: h is of type 1 if the number $\exp\{(2\pi m)/h\}$ is irrational for all $m \in \mathbb{Z} \setminus \{0\}$, and is of type 2 if it is not of type 1.

Define the set

$$\Omega = \prod_{p\in\mathbb{P}} \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for all $p \in \mathbb{P}$. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω, by the Tikhonov theorem, is a compact topological Abelian group. Denote by Ω_h the closed subgroup generated by the element $(p^{-ih} : p \in \mathbb{P})$, $h > 0$. Then it is known [3], see also [6], that

$$\Omega_h = \begin{cases} \Omega & \text{if } h \text{ is of type 1,} \\ \{\omega \in \Omega : \omega(a) = \omega(b)\} & \text{if } h \text{ is of type 2.} \end{cases}$$

Here

$$\frac{a}{b} = \exp\left\{\frac{2\pi m_0}{h}\right\}, \quad (a,b) = 1,$$

where $m_0 \in \mathbb{N}$ is the smallest number such that $\exp\{(2\pi m_0)/h\}$ is a rational number,

$$\omega(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p)$$

for $m \in \mathbb{N}$, and $\omega(p)$ denotes the pth component of $\omega \in \Omega$.

On $(\Omega_h, \mathcal{B}(\Omega_h))$, the probability Haar measure m_H^h can be defined, and we obtain the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$. Denote by $\omega_h(p)$ the pth component of an element

$\omega_h \in \Omega_h$, $p \in \mathbb{P}$, and, on the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$, define the $H(D)$-valued random element

$$\zeta(s, \omega_h) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\omega_h(p)}{p^s}\right)^{-1}.$$

We note that the latter infinite product, see, for example, ([7], Chapter 5), is uniformly convergent on compact subsets of the strip D for almost all ω_h with respect to the measure m_H^h.

Now, define the Dirichlet series which will be used for approximation of analytic functions. Let $\theta > 1/2$, $u > 0$, and

$$v_u(m) = \exp\left\{-\left(\frac{m}{u}\right)^\theta\right\}, \quad m \in \mathbb{N}.$$

Define

$$\zeta_u(s) = \sum_{m=1}^{\infty} \frac{v_u(m)}{m^s}.$$

Clearly, the latter series is absolutely convergent in the half-plane $\sigma > 1/2$. The main result of the paper is the following theorem of universality type. In what follows, $u = u_N$ in the definition of $\zeta_u(s)$.

Theorem 3. *Suppose that $u_N \to \infty$ and $u_N \ll N^2$ as $N \to \infty$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $h > 0$, the limit*

$$\lim_{N \to \infty} \frac{1}{N+1} \#\left\{0 \leqslant k \leqslant N : \sup_{s \in K} |\zeta_{u_N}(s + ikh) - f(s)| < \varepsilon\right\}$$
$$= m_H^h\left\{\omega_h \in \Omega_h : \sup_{s \in K} |\zeta(s, \omega_h) - f(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For a proof of Theorem 3, we apply a new probabilistic idea based on using of known universality theorems and application the machinery of distribution and characteristic functions. This way earlier was not known, and works well for a class of Dirichlet series which are close to universal functions. This method allows an extension of the class of universal functions, and has no connection to known methods. Thus, a proof of Theorem 3 is based on Theorem 2 and the estimate in the mean of the distance between $\zeta(s)$ and $\zeta_{u_N}(s)$.

3. Proof of Theorem 2

As mentioned above, the main ingredient of the probabilistic Bagchi method [3] for proof of universality theorems for Dirichlet series are limit theorems on weakly convergent probability measures in the space of analytic functions. We recall a result of discrete type from [3]. For $A \in \mathcal{B}(H(D))$, define

$$P_{N,h}(A) = \frac{1}{N+1} \#\{0 \leqslant k \leqslant N : \zeta(s + ikh) \in A\}.$$

Denote by $P_{\zeta,h}$ the distribution of $\zeta(s, \omega_h)$, i.e.,

$$P_{\zeta,h}(A) = m_H^h\{\omega_h \in \Omega_h : \zeta(s, \omega_h) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

Lemma 1. *For every $h > 0$, $P_{N,h}$ converges weakly to $P_{\zeta,h}$ as $N \to \infty$. Moreover, the support of the measure $P_{\zeta,h}$ is the set*

$$S \stackrel{def}{=} \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

A proof of the continuous version of Lemma 1 on the weak convergence for

$$\frac{1}{T}\text{meas}\{\tau \in [0,T] : \zeta(s+i\tau) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

as $T \to \infty$, apart [3], can be found, for example in [7,8]. The discrete case is considered [3] using a similar way. First it is proved that

$$\frac{1}{N+1}\#\Big\{0 \leqslant k \leqslant N : \Big(p^{-ikh} : p \in \mathbb{P}\Big) \in A\Big\}, \quad A \in \mathcal{B}(\Omega_h),$$

converges weakly to m_H^h as $N \to \infty$. This is applied to obtain that

$$\frac{1}{N+1}\#\{0 \leqslant k \leqslant N : \zeta_n(s+i\tau) \in A\}$$

and

$$\frac{1}{N+1}\#\{0 \leqslant k \leqslant N : \zeta_n(s+i\tau, \omega_h) \in A\}$$

$A \in \mathcal{B}(H(D))$, converge weakly to the same probability measure on $(H(D), \mathcal{B}(H(D)))$ as $N \to \infty$. Here, for $n \in \mathbb{N}$,

$$\zeta_n(s, \omega_h) = \sum_{m=1}^{\infty} \frac{v_n(m)\omega_h(m)}{m^s}$$

with

$$\omega_h(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_h^l(p).$$

The next step of the proof of Lemma 1 consists of the equalities

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |\zeta(s+ikh) - \zeta_n(s+ikh)| = 0$$

and, for almost all $\omega_h \in \Omega_h$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |\zeta(s+ikh, \omega_h) - \zeta_n(s+ikh, \omega_h)| = 0,$$

where K is an arbitrary compact subset of the strip D. Using the above facts and properties of weak convergence, it is obtained that $P_{N,h}$ and

$$\frac{1}{N+1}\#\{0 \leqslant k \leqslant N : \zeta(s+ikh, \omega_h) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

converge weakly to the same probability measure P_h on $(H(D), \mathcal{B}(H(D)))$ as $N \to \infty$. This and the Birkhoff–Khintchine ergodicity theorem connecting the arithmetic mean for ergodic processes with integrable sample paths to their expectation, see, for example [9], allows the detection of $P_h = P_{\zeta,h}$. For investigation of the support of $P_{\zeta,h}$, elements of the theory of exponential functions are applied.

For convenience, we recall the Mergelyan theorem on the approximation of analytic functions by polynomials, and equivalent of weak convergence of probability measures in terms of continuity sets.

Lemma 2. *Suppose that $K \subset \mathbb{C}$ is a compact set with connected complement, and $g(s)$ is a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p_\varepsilon(s)$ such that*

$$\sup_{s\in K} |g(s) - p_\varepsilon(s)| < \varepsilon.$$

Proof of the lemma can be found in ([10], Chapter 1).

A set $A \in \mathcal{B}(\mathbb{X})$ is called a continuity set of a probability measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ if $P(\partial A) = 0$, where ∂A denotes the boundary of A.

Lemma 3. *Suppose that P and P_n, $n \in \mathbb{N}$, are probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then P_n converges weakly to P if and only if*

$$\lim_{n\to\infty} P_n(A) = P(A)$$

for every continuity set A of the measure P.

Proof of the lemma is given, for example, in [11], Theorem 2.1.

Proof of Theorem 2. Define the set

$$A_\varepsilon = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\}.$$

The boundary ∂A_ε lies in the set

$$\left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon \right\}.$$

Therefore, the boundaries $\partial A_{\varepsilon_1}$ and $\partial A_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. From this, it follows that there are at most $m - 1$ values of ε such that $P_{\zeta,h}(\partial A_\varepsilon) > 1/m$, $m \in \mathbb{N}$, $m > 2$. Thus, $P_{\zeta,h}(\partial A_\varepsilon) > 0$ for at most countably many $\varepsilon > 0$. In other words, the set A_ε is a continuity set of the measure $P_{\zeta,h}$ for all but at most countably many $\varepsilon > 0$. Therefore, in view of Lemmas 1 and 3,

$$\lim_{N\to\infty} P_{N,h}(A_\varepsilon) = P_{\zeta,h}(A_\varepsilon)$$

for all but at most countably many $\varepsilon > 0$. The definitions of $P_{N,h}$, $P_{\zeta,h}$ and A_ε show that

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s + ikh) - f(s)| < \varepsilon \right\}$$
$$= m_H^h \left\{ \omega_h \in \Omega_h : \sup_{s\in K} |\zeta(s, \omega_h) - f(s)| < \varepsilon \right\}$$

for all but at most countably many $\varepsilon > 0$. It remains to prove that the right-hand side of the latter equality is strongly positive.

Since $f(s) \neq 0$ on K, the principal branch of logarithm $\log f(s)$ satisfies hypotheses of Lemma 2. Therefore, for every $\delta > 0$, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |\log f(s) - p(s)| < \delta.$$

Hence, using the well-known inequality $|\mathrm{e}^z - 1| \leqslant |z|\mathrm{e}^{|z|}$ which is valid for all $z \in \mathbb{C}$, gives

$$\left| f(s) - \mathrm{e}^{p(s)} \right| = \left| \mathrm{e}^{\log f(s)} - \mathrm{e}^{p(s)} \right| = |f(s)| \left| \mathrm{e}^{p(s) - \log f(s)} - 1 \right|$$
$$\leqslant |f(s)| |\log f(s) - p(s)| \mathrm{e}^{|\log f(s) - p(s)|} \leqslant \sup_{s\in K} |f(s)| \delta \mathrm{e}^\delta \quad (1)$$

for all $s \in K$. Without loss of generality, we may suppose that $0 < \varepsilon < 1$. Let $M = \max(\sup_{s\in K} |f(s)|, 1)$, and

$$\delta = \frac{\varepsilon}{2M\mathrm{e}}.$$

Then (1) implies the inequality

$$\sup_{s\in K}\left|f(s)-\mathrm{e}^{p(s)}\right|<\frac{\varepsilon}{2}. \tag{2}$$

Define the set

$$G_\varepsilon=\left\{g\in H(D):\sup_{s\in K}\left|g(s)-\mathrm{e}^{p(s)}\right|<\frac{\varepsilon}{2}\right\}.$$

Suppose that $g\in G_\varepsilon$. Then, in virtue of (2),

$$\sup_{s\in K}|g(s)-f(s)|\leqslant\sup_{s\in K}\left|g(s)-\mathrm{e}^{p(s)}\right|+\sup_{s\in K}\left|f(s)-\mathrm{e}^{p(s)}\right|<\varepsilon,$$

i.e., $g\in A_\varepsilon$. Thus

$$G_\varepsilon\subset A_\varepsilon. \tag{3}$$

Since $\mathrm{e}^{p(s)}\in S$, by the second part of Lemma 1, the function $\mathrm{e}^{p(s)}$ is an element of the support of the measure $P_{\zeta,h}$. Therefore, the set G_ε is an open neighborhood of an element of the support of $P_{\zeta,h}$. Hence, by the properties of the support,

$$P_{\zeta,h}(G_\varepsilon)>0,$$

and the inclusion (3) shows that $P_{\zeta,h}(A_\varepsilon)>0$. The theorem is proved. □

4. Distance between $\zeta(s)$ and $\zeta_{u_N}(s)$

We will derive Theorem 3 from Theorem 2. For this, we will use the closeness between $\zeta(s)$ and $\zeta_{u_N}(s)$. First, we recall the Gallagher lemma, see Lemma 1.4 of [12], which connects discrete and continuous mean squares of some differentiable functions.

Lemma 4. *Suppose that T_0, $T\geqslant\delta>0$ and $\mathcal{T}\neq\varnothing$ is a finite set in the interval $[T_0+\delta/2,T_0+T-\delta/2]$. Define*

$$N_\delta(x)=\sum_{\substack{t\in\mathcal{T}\\|t-x|<\delta}}1,$$

and suppose that $S(x)$ is a complex-valued continuous function on $[T_0,T_0+T]$ with a continuous derivative on (T_0,T_0+T). Then

$$\sum_{t\in\mathcal{T}}N_\delta^{-1}(t)|S(t)|^2\leqslant\frac{1}{\delta}\int\limits_{T_0}^{T_0+T}|S(x)|^2\,\mathrm{d}x+\left(\int\limits_{T_0}^{T_0+T}|S(x)|^2\,\mathrm{d}x\int\limits_{T_0}^{T_0+T}|S'(x)|^2\,\mathrm{d}x\right)^{1/2}.$$

Lemma 5. *Suppose that $u_N\to\infty$ and $u_N\ll N^2$ as $N\to\infty$. Then*

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|\zeta(s+ikh)-\zeta_{u_N}(s+ikh)|=0.$$

Proof. Let, as usual, $\Gamma(s)$ denote the Euler gamma-function, and θ be from the definition of $v_u(m)$. Then an application of the Mellin formula

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\Gamma(s)a^{-s}\,\mathrm{d}s=\mathrm{e}^{-a},\quad a,b>0,$$

shows that

$$v_u(m)=\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\frac{1}{\theta}\Gamma\left(\frac{s}{\theta}\right)\left(\frac{m}{u}\right)^{-s}\mathrm{d}s.$$

Since the series for $\zeta_u(s)$ is absolutely convergent, putting

$$l_u(s) = \frac{s}{\theta}\Gamma\Big(\frac{s}{\theta}\Big)u^s$$

gives, for $\sigma > 1/2$,

$$\begin{aligned}\zeta_u(s) &= \sum_{m=1}^{\infty}\frac{1}{m^s}\bigg(\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\frac{z}{\theta}\Gamma\Big(\frac{z}{\theta}\Big)\Big(\frac{m}{u}\Big)^{-z}\frac{\mathrm{d}z}{z}\bigg) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\bigg(\sum_{m=1}^{\infty}\frac{1}{m^{s+z}}\frac{l_u(z)}{z}\bigg)\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z)l_u(z)\frac{\mathrm{d}z}{z}.\end{aligned}$$

Let $\theta_1 > 0$. Then, by the residue theorem,

$$\zeta_{u_N}(s) - \zeta(s) = \frac{1}{2\pi i}\int_{\theta_1-i\infty}^{\theta_1+i\infty}\zeta(s+z)l_{u_N}(z)\frac{\mathrm{d}z}{z} + R_{u_N}(s), \tag{4}$$

where

$$R_{u_N}(s) = \frac{l_{u_N}(1-s)}{1-s}.$$

There exists $\varepsilon > 0$ such that $1/2 + 2\varepsilon \leqslant \sigma \leqslant 1-\varepsilon$ for all $s \in K$. Then $\sigma - 1/2 - \varepsilon \overset{def}{=} \theta_1 > 0$, and, in view of (4), we have, for $s \in K$,

$$\begin{aligned}\zeta_{u_N}(s+ikh) - \zeta(s+ikh) =& \frac{1}{2\pi i}\int_{-\infty}^{\infty}\zeta\Big(\sigma + it - \sigma + \frac{1}{2} + \varepsilon + ikh + i\tau\Big)\frac{l_{u_N}(1/2+\varepsilon-\sigma+i\tau)}{1/2+\varepsilon-\sigma+i\tau}\,\mathrm{d}\tau \\ &+ R_{u_N}(s+ikh).\end{aligned}$$

Hence, shifting $t+\tau$ to τ, we obtain, for $s \in K$,

$$\begin{aligned}\zeta_{u_N}(s+ikh) - \zeta(s+ikh) =& \frac{1}{2\pi i}\int_{-\infty}^{\infty}\zeta\Big(\frac{1}{2} + \varepsilon + ikh + i\tau\Big)\frac{l_{u_N}(1/2+\varepsilon-s+i\tau)}{1/2+\varepsilon-s+i\tau}\,\mathrm{d}\tau + R_{u_N}(s+ikh) \\ &\ll \int_{-\infty}^{\infty}\Big|\zeta\Big(\frac{1}{2}+\varepsilon+ikh+i\tau\Big)\Big|\sup_{s\in K}\frac{|l_{u_N}(1/2+\varepsilon-s+i\tau)|}{|1/2+\varepsilon-s+i\tau|}\,\mathrm{d}\tau \\ &+ \sup_{s\in K}|R_{u_N}(s+ikh)|.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|\zeta(s+ikh) - \zeta_{u_N}(s+ikh)| \ll& \int_{-\infty}^{\infty}\bigg(\frac{1}{N+1}\sum_{k=0}^{N}\Big|\zeta\Big(\frac{1}{2}+\varepsilon+ikh+i\tau\Big)\Big|\bigg) \\ &\times\sup_{s\in K}\frac{|l_{u_N}(1/2+\varepsilon-s+i\tau)|}{|1/2+\varepsilon-s+i\tau|}\,\mathrm{d}\tau \\ &+ \frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|R_{u_N}(s+ikh)| \\ &\overset{def}{=} I(N) + Z(N).\end{aligned} \tag{5}$$

It is well known that for fixed σ, $1/2 < \sigma < 1$,

$$\int_{-T}^{T}|\zeta(\sigma+it)|^2\,\mathrm{d}t \ll_\sigma T, \quad \text{and} \quad \int_{-T}^{T}|\zeta'(\sigma+it)|^2\,\mathrm{d}t \ll_\sigma T,$$

where $\ll_\sigma$ means that the implied constant depends on σ. Hence, for all real τ,

$$\int_{-T}^{T}|\zeta(\sigma+i\tau+it)|^2\,\mathrm{d}t \ll_\sigma T(1+|\tau|), \quad \text{and} \quad \int_{-T}^{T}|\zeta'(\sigma+i\tau+it)|^2\,\mathrm{d}t \ll_\sigma T(1+|\tau|). \tag{6}$$

Lemma 4 with $\delta = h$, $T_0 = h$, $\tau = Nh$, $\mathcal{T} = \{2h, \dots, Nh\}$ and (6) implies the estimate

$$\sum_{k=0}^{N} |\zeta(\sigma + ikh + i\tau)|^2 \ll_{\sigma,h} |\tau| + \frac{1}{h}\int_0^{Nh} |\zeta(\sigma + i\tau + it)|^2 \, \mathrm{d}t + \left(\int_0^{Nh} |\zeta(\sigma + i\tau + it)|^2 \, \mathrm{d}t \int_0^{Nh} |\zeta'(\sigma + i\tau + it)|^2 \, \mathrm{d}t\right)^{1/2} \ll_{\sigma,h} N(1 + |\tau|).$$

From this, we find

$$\frac{1}{N+1}\sum_{k=0}^{N}\left|\zeta\left(\frac{1}{2} + \varepsilon + ikh + i\tau\right)\right| \ll \left(\frac{1}{N+1}\sum_{k=0}^{N}\left|\zeta\left(\frac{1}{2} + \varepsilon + ikh + i\tau\right)\right|^2\right)^{1/2} \ll_{\varepsilon,h} (1 + |\tau|)^{1/2} \ll_{\varepsilon,h} 1 + |\tau|. \tag{7}$$

It is well known that uniformly in σ, $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, with arbitrary $\sigma_1 < \sigma_2$,

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}. \tag{8}$$

Therefore, by definition of l_{u_N}, for $s \in K$,

$$\frac{l_{u_N}(1/2 + \varepsilon - s + i\tau)}{1/2 + \varepsilon - s + i\tau} \ll_\theta u_N^{1/2+\varepsilon-\sigma}\left|\Gamma\left(\frac{1}{\theta}\left(\frac{1}{2} + \varepsilon - it + i\tau\right)\right)\right| \ll_\theta u_N^{-\varepsilon}\exp\left\{-\frac{c}{\theta}|\tau - t|\right\} \ll_{\theta,K} u_N^{-\varepsilon}\exp\{-c_1|\tau|\}, \quad c_1 > 0.$$

This and (7) show that

$$I(N) \ll_{\varepsilon,h,\theta,K} u_N^{-\varepsilon}\int_{-\infty}^{\infty}(1 + |\tau|)^{1/2}\exp\{-c_1|\tau|\}\,\mathrm{d}\tau \ll_{\varepsilon,h,\theta,K} u_N^{-\varepsilon}. \tag{9}$$

Using (8), we obtain similarly that

$$Z(N) \ll_\theta \frac{u_N^{1-\sigma}}{N}\sum_{k=0}^{N}\exp\left\{-\frac{c}{\theta}|kh - t|\right\} \ll_{\theta,K} \frac{u_N^{1/2-2\varepsilon}}{N}\sum_{k=0}^{N}\exp\{-c_2hk\} \ll_{\theta,h,K} u_N^{1/2-2\varepsilon}\left(\frac{u_N^{\varepsilon}}{N} + \frac{\exp\{-c_2hu_N^{\varepsilon}\}}{N}\right), \quad c_2 > 0.$$

This estimate together with (9) and (5) proves the lemma. □

5. Proof of Theorem 3

We will apply the method of characteristic functions and the weak convergence of distribution functions. Recall that every real non-decreasing left continuous function $F(x)$, $x \in \mathbb{R}$, such that $F(+\infty) = 1$ and $F(-\infty) = 0$ is called the distribution function. It has at most countably many discontinuity points.

Let $F(x)$ and $F_n(x)$, $n \in \mathbb{N}$, be distribution functions. We say that $F_n(x)$ converges weakly to $F(x)$ as $n \to \infty$ if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for all continuity points x of $F(x)$.

Denote by $\varphi(u)$ the characteristic function of a distribution function $F(x)$, i.e.,

$$\varphi(u) = \int_{-\infty}^{\infty} \mathrm{e}^{iux}\,\mathrm{d}F(x).$$

The weak convergence of distribution functions and convergence of the corresponding characteristic functions are connected by continuity theorems, see, for example, [13].

Lemma 6. *Suppose that $F_n(x)$ converges weakly to $F(x)$ as $n \to \infty$. Then $\varphi_n(u)$ converges to $\varphi(u)$ as $n \to \infty$ uniformly in u in every finite interval.*

Lemma 7. *Suppose that $\varphi_n(u)$ as $n \to \infty$ converges to a function $\varphi(u)$ continuous at the point $u = 0$. Then $F_n(x)$ converges weakly to a certain distribution function $F(x)$ as $n \to \infty$. The function $\varphi(u)$ is the characteristic function of $F(x)$.*

Proof of Theorem 3. Let A_ε be the same as in the proof of Theorem 2. For the objects of Theorem 3, define the functions

$$F_{N,h}(\varepsilon) = P_{N,h}(A_\varepsilon) = \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon\right\},$$

$$F_{N,u_N,h}(\varepsilon) = \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta_{u_N}(s+ikh;\mathfrak{a}) - f(s)| < \varepsilon\right\}$$

and

$$F_{\zeta,h}(\varepsilon) = P_{\zeta,h}(A_\varepsilon) = m_H^h\left\{\omega_h \in \Omega_h : \sup_{s\in K} |\zeta(s,\omega_h) - f(s)| < \varepsilon\right\}.$$

Then the functions $F_{N,h}(\varepsilon)$, $F_{N,u_N,h}(\varepsilon)$ and $F_{\zeta,h}(\varepsilon)$ with respect to ε are distribution functions. Moreover, the above definitions imply the equality

$$\begin{aligned}
P_{\zeta,h}(\partial A_\varepsilon) &= P_{\zeta,h}\left\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon\right\} \\
&= P_{\zeta,h}\left\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| \leqslant \varepsilon\right\} - P_{\zeta,h}\left\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\right\} \\
&= m_H^h\left\{\omega_h \in \Omega_h : \sup_{s\in K} |\zeta(s,\omega_h) - f(s)| \leqslant \varepsilon\right\} - m_H^h\left\{\omega_h \in \Omega_h : \sup_{s\in K} |\zeta(s,\omega_h) - f(s)| < \varepsilon\right\} \\
&= F_{\zeta,h}(\varepsilon+0) - F_{\zeta,h}(\varepsilon).
\end{aligned}$$

Therefore, $P_{\zeta,h}(\partial A_\varepsilon) = 0$ if and only if $F_{\zeta,h}(\varepsilon+0) = F_{\zeta,h}(\varepsilon)$, i.e., if ε is a continuity point of $F_{\zeta,h}$. By Lemmas 1 and 3, $P_{N,h}(A_\varepsilon)$ converges to $P_{\zeta,h}(A_\varepsilon)$ as $N \to \infty$ for ε such that $P_{\zeta,h}(\partial A_\varepsilon) = 0$. Hence, $F_{N,h}(\varepsilon)$ converges weakly to $F_{\zeta,h}(\varepsilon)$ as $N \to \infty$. This and Lemma 6 show the convergence of the corresponding characteristic functions, i.e.,

$$\varphi_{N,h}(u) = \varphi_{\zeta,h}(u) + o(1) \tag{10}$$

as $N \to \infty$ uniformly in u, $|u| \leqslant C$ with every fixed $C > 0$. It remains to prove that the above asymptotic equality remains valid with $\varphi_{N,u_N,h}(u)$ in place of $\varphi_{N,h}(u)$. Actually, we have

$$\varphi_{N,u_N,h}(u) = \int_{-\infty}^{\infty} \mathrm{e}^{iu\varepsilon}\,\mathrm{d}F_{N,u_N,h}(\varepsilon) = \varphi_{N,h}(\varepsilon) + \int_{-\infty}^{\infty} \mathrm{e}^{iu\varepsilon}\,\mathrm{d}(F_{N,u_N,h}(\varepsilon) - F_{N,h}(\varepsilon)).$$

Therefore, in view of the definitions of $F_{N,h}(\varepsilon)$ and $F_{N,u_N,h}(\varepsilon)$, using the inequality $|\mathrm{e}^{iu} - 1| \leqslant |u|$, $u \in \mathbb{R}$, as well as the triangle inequality

$$\left|\sup_s |g_1(s) - g(s)| - \sup_s |g_2(s) - g(s)|\right| \leqslant \sup_s |g_1(s) - g_2(s)|, \quad \forall g_1, g_2, g,$$

we obtain

$$
\begin{aligned}
&|\varphi_{N,h}(\varepsilon) - \varphi_{N,u_N,h}(\varepsilon)| \\
&= \left| \frac{1}{N+1} \sum_{k=0}^{N} \exp\left\{ iu \sup_{s\in K} |\zeta(s+ikh) - f(s)| \right\} - \frac{1}{N+1} \sum_{k=0}^{N} \exp\left\{ iu \sup_{s\in K} |\zeta_{u_N}(s+ikh) - f(s)| \right\} \right| \\
&\leqslant \frac{1}{N+1} \sum_{k=0}^{N} \left| \exp\left\{ iu \left(\sup_{s\in K} |\zeta(s+ikh) - f(s)| - \sup_{s\in K} |\zeta_{u_N}(s+ikh) - f(s)| \right) \right\} - 1 \right| \\
&\leqslant \frac{|u|}{N+1} \sum_{k=0}^{N} \left| \sup_{s\in K} |\zeta(s+ikh) - f(s)| - \sup_{s\in K} |\zeta_{u_N}(s+ikh) - f(s)| \right| \\
&\leqslant \frac{|u|}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |\zeta(s+ikh) - \zeta_{u_N}(s+ikh)|.
\end{aligned}
$$

Hence, by Lemma 5, we have

$$g_{N,h}(u) = g_{N,u_N,h}(u) + o(1)$$

as $N \to \infty$ uniformly in $|u| \leqslant C$. Thus, in view of (10),

$$g_{N,u_N,h}(u) = g_{\zeta,h}(u) + o(1)$$

as $N \to \infty$ uniformly in $|u| \leqslant C$. The function $g_{\zeta,h}(u)$ is a continuous, therefore, Lemma 7, implies that $F_{N,u_N,h}(\varepsilon)$ converges weakly to $F_{\zeta,h}(\varepsilon)$ as $N \to \infty$, and the theorem is proved because $F_{\zeta,h}(\varepsilon)$ has at most countably many discontinuity points. □

6. Conclusions

Universality of zeta-functions and Dirichlet series, in general, is a very interesting phenomenon of the theory of analytic functions. Roughly speaking, universality means that a wide class of analytic functions can be approximated by shifts of one and the same Dirichlet series (in the case of the Riemann zeta-function, by shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$). Clearly, for practical applications, the use of discrete shifts $\zeta(s+ikh)$, $h > 0$, is more convenient than $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$, because it is easier to find k in the set $\mathbb{N}_0$ than τ in the set $\mathbb{R}$ with approximating property. Moreover, it is more convenient to deal with absolutely convergent series. In the paper, we propose the discrete approximation of analytic functions by shifts $\zeta_{u_N}(s+ikh)$, where $\zeta_{u_N}(s)$ is a certain sequence of absolutely convergent Dirichlet series. From the main theorem (Theorem 3), it follows that there exist $N_0 = N_0(f,\varepsilon,K,h) > 0$ and $c > 0$ such that for every $N \geqslant N_0(f,\varepsilon,K,h)$, there are more than cN shifts $\zeta_{u_N}(s+ikh)$ such that

$$\sup_{s\in K} |\zeta_{u_N}(s+ikh) - f(s)| < \varepsilon$$

for $K \in \mathcal{K}$ and $f(s) \in H_0(K)$.

Author Contributions: Conceptualization, A.L. and D.Š.; Methodology, A.L. and D.Š.; Investigation, A.L. and D.Š.; Writing–original draft preparation, A.L. and D.Š.; Writing–review and editing, A.L. and D.Š. All authors have read and agreed to the published version of the manuscript.

Funding: The research of the first author is funded by the European Social Fund (project No. 09.3.3-LMT-K-712-01-0037) under grant agreement with the Research Council of Lithuania (LMT LT).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Voronin, S.M. Theorem on the "universality" of the Riemann zeta-function. *Math. USSR Izv.* **1975**, *9*, 443–453. [CrossRef]
2. Reich, A. Werteverteilung von Zetafunktionen. *Arch. Math.* **1980**, *34*, 440–451. [CrossRef]
3. Bagchi, B. The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and Other Allied Dirichlet Series. Ph. D. Thesis, Indian Statistical Institute, Calcutta, India, 1981.
4. Pańkowski, Ł. Joint universality for dependent L-functions. *Ramanujan J.* **2018**, *45*, 181–195. [CrossRef]
5. Mauclaire, J.-L. Universality of the Riemann zeta-function: Two remarks. *Ann. Univ. Sci. Budapest. Sect. Comp.* **2013**, *39*, 311–319.
6. Laurinčikas, A., Matsumoto, K., Steuding, J. Discrete universality of L-functions for new forms. II. *Lith. Math. J.* **2016**, *56*, 207–218. [CrossRef]
7. Laurinčikas, A. *Limit Theorems for the Riemann Zeta-Function*; Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 1996.
8. Steuding, J. *Value-Distribution of L-Functions*; Lecture Notes Math; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2007; Volume 1877.
9. Cramér, H., Leadbetter, M.R. *Stationary and Related Stochastic Processes*; Willey: New York, NY, USA, 1967.
10. Mergelyan, S.N. Uniform approximations to functions of a complex variable. *Am. Math. Soc. Transl.* **1962**, *3*, 294–391.
11. Billingsley, P. *Convergence of Probability Measures*; Willey: New York, NY, USA, 1968.
12. Montgomery, H.L. *Topics in Multiplicative Number Theory*; Lecture Notes Math; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1971; Volume 227.
13. Gnedenko, B.V. *Theory of Probability*, 6th ed.; CRC Press: Boca Raton, FL USA; London, UK; New York, NY, USA, 1997.